

DRINFELD REALIZATION OF AFFINE QUANTUM ALGEBRAS: THE RELATIONS.

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*To women.
Especially to those who do not have
even the opportunity to imagine
how much they would like mathematics
and to those who are forced to forget it.*

ABSTRACT. In this paper the structure of the Drinfeld realization \mathcal{U}_q^{Dr} of affine quantum algebras (both untwisted and twisted) is described in details, and its defining relations are studied and simplified. As an application, a homomorphism ψ from this realization to the Drinfeld and Jimbo presentation \mathcal{U}_q^{DJ} is provided, and proved to be surjective.

§0. INTRODUCTION.

Let $X_{\tilde{n}}^{(k)}$ be a Dynkin diagram of affine type, $\mathcal{U}_q^{DJ} = \mathcal{U}_q^{DJ}(X_{\tilde{n}}^{(k)})$ the quantum algebra introduced by Drinfeld and Jimbo (see [Dr2] and [Jm]), $\mathcal{U}_q^{Dr} = \mathcal{U}_q^{Dr}(X_{\tilde{n}}^{(k)})$ its Drinfeld realization (see [Dr1]).

This paper has two main goals: describing in details the structure of the Drinfeld realization \mathcal{U}_q^{Dr} arriving at sharply simplifying its defining relations; and constructing a (surjective) homomorphism ψ from this realization to the Drinfeld and Jimbo presentation \mathcal{U}_q^{DJ} , as a step to provide a complete proof that \mathcal{U}_q^{DJ} and \mathcal{U}_q^{Dr} are isomorphic, so that they are indeed different presentations of the same $\mathbb{C}(q)$ -algebra $\mathcal{U}_q = \mathcal{U}_q(X_{\tilde{n}}^{(k)})$ (see [Dr1]).

Understanding the isomorphism between \mathcal{U}_q^{DJ} and \mathcal{U}_q^{Dr} stated by Drinfeld in [Dr1] has important applications in the study of the representation theory of affine quantum algebras: using this result the finite dimensional irreducible representations of affine quantum algebras are classified in [CP1], [CP2] and [CP3]; a geometrical realization (through the quiver varieties) of finite dimensional representations is constructed in [N] for the untwisted simply laced cases.

The interest of the twisted case resides not only in that it is a generalization of the untwisted frame. Actually twisted algebras appear quite naturally while studying the untwisted setting, due to the fact that the transposition of matrices establishes a duality among the affine Cartan matrices through which untwisted Cartan matrices can correspond to twisted ones; more precisely simply laced untwisted matrices and the matrices of type $A_{2n}^{(2)}$ are self-dual, while transposition operates on the remaining affine Cartan matrices by interchanging untwisted and twisted ones. This observation is important and concrete because of results like those in [CP4], where the quantum symmetry group of the affine Toda field theory associated to an untwisted affine Kac-Moody algebra is proved to be the quantum algebra associated to the dual Kac-Moody algebra; and in [FH], where the authors conjecture in general, and prove for the Kirillov-Reshetikhin modules, that there exists a duality between the representations of an untwisted affine quantum algebra and those of the dual quantum algebra.

Much work has already been done in the direction of understanding Drinfeld's theorem: in [Be] all the relations are proved in the untwisted case. Notice that this does not yet imply that ψ is an isomorphism: indeed the argument for the injectivity should be completed with the proof of the existence of a basis of the integer form, necessary to conclude that the injectivity at 1 implies the injectivity at q -level; this point is not discussed and as far as I understand non-trivial.

For the twisted case there are several partial results: in [A] the author studies the case $A_2^{(2)}$, constructing ψ following [Be], but the proof that it is well defined is incomplete; a contribution to this proof is given in [H].

In [Jn], [JZ2] and [JZ1], the authors construct a homomorphism from $\mathcal{U}_q^{DJ}(X_{\tilde{n}}^{(k)})$ to $\mathcal{U}_q^{Dr}(X_{\tilde{n}}^{(k)})$ (the inverse of ψ) following the theorem stated by Drinfeld in [Dr1], that is by means of the q -commutators. In [Jn] the author gives some details in the untwisted case, sketching the proof of relations $[E_0, F_i] = 0$ ($i \in I_0$) in case $A_3^{(1)}$, of the Serre relation $E_0 E_1^2 - (q + q^{-1}) E_1 E_0 E_1 + E_1^2 E_0 = 0$ in case $A_n^{(1)}$ (noticing that the Serre relations involving just indices in I_0 are trivial, but the other Serre relations involving E_0 are not studied, for instance $E_1 E_0^2 - (q + q^{-1}) E_0 E_1 E_0 + E_0^2 E_1 = 0$ is missing) and of relations $[E_0, F_0] = \frac{K_0 - K_0^{-1}}{q_0 - q_0^{-1}}$ in cases $A_n^{(1)}$ and $C_2^{(1)}$; but a strategy for generalizing these arguments is not presented, and the twisted case is just stated to be similar. In [JZ2] the authors concentrate on the twisted case, but their work is

again incomplete since the Serre relations involving indices $i \neq j \in I_0$ are treated, erroneously, as in the untwisted case, and for the other relations the authors present some examples: the commutation between E_0 and F_i ($i \in I$) is studied in cases $A_2^{(2)}$ and $D_4^{(3)}$; some Serre relations (but not all of them) involving E_0 are studied in the cases $A_{2n-1}^{(2)}$ and $D_4^{(3)}$; and again a strategy for generalizing these computations is not shown. It should be noticed a mistake in the connection between the data of the finite Dynkin diagram and its non trivial automorphism on one hand and the twisted affine Dynkin diagram on the other hand, which has consequences in the following paper [JZ1]. Finally in [JZ1] the authors want to fill the gap about the Serre relations involving the indices $i, j \in I_0$ such that $a_{ij} < -1$ (in the twisted case), and they use a case by case approach: but the Drinfeld relations are misunderstood, and stated to imply relations not holding in this algebra.

These difficulties suggest the need to pay some care in understanding the Drinfeld realization, which is the aim of the present paper: the definition of the homomorphism ψ from the Drinfeld realization to the Drinfeld and Jimbo presentation of affine quantum algebras becomes then a simple consequence of this analysis, and that ψ is surjective is also proved.

In sections §1 and §2 we recall the notions of Dynkin diagram, Weyl group and root system, and their properties needed in developing the arguments of the following sections: in particular it is recalled how untwisted and twisted affine Dynkin diagrams, Weyl groups and root systems are connected to finite ones, and their classification and basic properties.

Section §3 is again a section where some preliminary material (about the presentation of Drinfeld and Jimbo of the (affine) quantum algebras \mathcal{U}_q^{DJ}) is summarized.

In definitions 2 and 3 and in remark 4 we recall the definition of \mathcal{U}_q^{DJ} , its main structures (the Q -gradation, the triangular decomposition, the antiautomorphisms Ω and Ξ , the braid group action, the embedding of the finite quantum algebra in the affine one, the root vectors E_α) and properties (commutation among the (anti)automorphisms, connection between braid group action and root vectors, Poincaré-Birkhoff-Witt basis, Levendorskii-Soibelman formula).

We recall also the embeddings φ_i of the rank 1 quantum algebras $\mathcal{U}_q^{DJ}(A_1^{(1)})$ or $\mathcal{U}_q^{DJ}(A_2^{(2)})$ in the general quantum algebra $\mathcal{U}_q^{DJ}(X_n^{(k)})$ and

their properties of commutation and injectivity (definition 6 and remark 7). They will play a role in the comparison between the Drinfeld realization and the Drinfeld-Jimbo presentation of section §12 (theorem §12.7).

In section §4 we give the definition of the Drinfeld realization of affine quantum algebras (both untwisted and twisted, see [Dr1]), discussing and translating the relations in a more explicit form, easier for the purpose of this paper. Even if it is just a reformulation, it seems useful to give the details, since they are not always clear in the literature.

In section §5 some notations are fixed in order to simplify the analysis of the relations. Also some relations are reformulated in terms of q -commutators, and some new relations as the Serre relations (S^\pm) and other similar ($(T2^\pm)$ and $(T3^\pm)$) are introduced, which will play an important role in the following sections (§10 and §11).

In section §6 the main structures on \mathcal{U}_q^{Dr} are introduced: the Q -gradation; the homomorphisms ϕ_i , underlining the role of the two affine Drinfeld realizations of rank one, $A_1^{(1)}$ and $A_2^{(2)}$, which embed in any other Drinfeld realization, each embedding depending on the choice of a vertex of the (“finite part” of the) Dynkin diagram; the antiautomorphism Ω , describing the correspondence between “positive” and “negative” vectors $X_{i,r}^\pm$; the automorphisms Θ and t_i (for each $i \in I_0$), which summarize several symmetries (reflection around zero and translations) among the “positive” vectors. Actually these structures are defined on the algebra $\bar{\mathcal{U}}_q^{Dr}$ (which is also defined in this section), of which the Drinfeld realization is a quotient, and the proof that they induce analogous structures on \mathcal{U}_q^{Dr} is quickly concluded in section §8, through the discussion of section §7.

In section §7 the algebra $\tilde{\mathcal{U}}_q^{Dr}$, which is an algebra (already introduced in the previous section) intermediate between $\bar{\mathcal{U}}_q^{Dr}$ and \mathcal{U}_q^{Dr} , is studied in details. In particular a first set of relations is simplified: the most important remarks are that the relations (HX^\pm) can be replaced by the much easier (HXL^\pm) , see proposition 15 (they are much easier not only because they are a smaller set of relations, but mainly because they can be expressed just in terms of q -commutation of the generators $X_{i,r}^\pm$ ’s of $\bar{\mathcal{U}}_q^{Dr}$, without using the $H_{i,r}$ ’s, see remark 18); and that the relations (HH) are also redundant, see proposition 16. But also the other relations are studied and interpreted while discussing how the structures on $\bar{\mathcal{U}}_q^{Dr}$ (see section §6) induce analogous structures on $\tilde{\mathcal{U}}_q^{Dr}$, see remarks 7 and 9.

Section §8 is a short and simple section where the structures defined on $\bar{\mathcal{U}}_q^{Dr}$ and induced on $\tilde{\mathcal{U}}_q^{Dr}$ are proved to pass also to \mathcal{U}_q^{Dr} ; this simple analysis is carried out explicitly, fixing some notations, in order to use it in further considerations, especially in section §9.

In section §9 it is now possible to start concentrating on the simplification of the relations defining \mathcal{U}_q^{Dr} over $\tilde{\mathcal{U}}_q^{Dr}$: these are the relations involving just the $X_{i,r}^+$'s or just the $X_{i,r}^-$'s, and there is a correspondence between the two cases thanks to the action of $\tilde{\Omega}$. The main result of this section is that the dependence of these relations on parameters $(r_1, \dots, r_l) \in \mathbb{Z}^l$ ($l \in \mathbb{Z}_+$) is redundant: we can indeed just restrict to the same relations indexed by $(r, \dots, r) \in \mathbb{Z}^l$ where $r \in \mathbb{Z}$ (the “constant parameter” relations), so that the dependence on \mathbb{Z}^l is reduced to a dependence on an integer r (see lemmas 12 and 14, proposition 15 and corollary 19); on the other hand, thanks to the action of the \tilde{t}_i 's, this situation can be again simplified just analyzing the relations relative to $(0, \dots, 0)$ (see remark 8).

Thanks to the results of section §9 the study of the relations defining \mathcal{U}_q^{Dr} can be pushed forward: in section §10 further dependences among the relations are proved (propositions 1 and 4, corollary 6 and remark 7). These results are summarized in theorem 8 and in corollary 9, where a “minimal” set of relations is provided.

The last step of this analysis is the study of the Serre relations, performed in section §11: here the relations $(XD^\pm)-(S3^\pm)$ are proved to depend, in the case of rank bigger than 1, on the (“constant parameter”) Serre relations, and these are viceversa proved to depend on the relations $(XD^\pm)-(S3^\pm)$ also in the cases in which it is not tautologically evident ($k > 1$, $a_{ij} < -1$). Theorem 18 and corollary 19 state the final result of this study, and are the main tool for constructing the homomorphism ψ and for proving that it is well defined, see section §12.

Section §12 is devoted to construct a homomorphism ψ from \mathcal{U}_q^{Dr} to \mathcal{U}_q^{DJ} and to prove that it is well defined and surjective.

In definition 3 $\tilde{\psi} : \tilde{\mathcal{U}}_q^{Dr} \rightarrow \mathcal{U}_q^{DJ}$ is defined, following [Be]. It just requires some care in the determination of the sign o (notation 1 and remark 2).

The results of section §11 and the correspondence, described in proposition 4, between the (anti)automorphisms constructed on \mathcal{U}_q^{Dr}

and those already known on \mathcal{U}_q^{DJ} make the goal of proving that $\tilde{\psi}$ induces ψ on \mathcal{U}_q^{Dr} trivial in the cases of rank bigger than 1, that is in all cases different from $A_1^{(1)}$ and $A_2^{(2)}$ (theorem 5).

We give two different arguments to solve the cases of rank one (theorem 7). The first one is based on the direct computation of the simple commutation relation between E_1 and $E_{\delta+\alpha_1}$ in $\mathcal{U}_q^{DJ}(A_1^{(1)})$ and $\mathcal{U}_q^{DJ}(A_2^{(2)})$ (lemma 6). The second one is a straightforward corollary of the result in the case of rank bigger than 1, once one recalls the embeddings (see remark §3.7) of the rank 1 quantum algebras in the general quantum algebras.

A proof that ψ is surjective is provided in theorem 11: it makes use of the correspondence between the automorphisms t_i on \mathcal{U}_q^{Dr} and the automorphisms T_{λ_i} on \mathcal{U}_q^{DJ} and among the Ω 's (remark 8), and of the braid group action on \mathcal{U}_q^{DJ} .

Theorem 11 would suggest also how to define the inverse of ψ .

An index of the notations used in the paper is listed in the appendix, section §13.

I'm deeply thankful to David Hernandez for proposing me to work again on the twisted affine quantum algebras: I abandoned them too many years ago, and would have neither planned nor dared to approach them again if he had not encouraged and motivated me.

I take this occasion to make explicit my gratitude to Corrado De Concini, my *maestro*: for his always caring presence (even when he did not approve my choices) in the vicissitudes of my relationship with mathematics, and for his belief (undeserved yet helpful) he made me always feel. Not accidentally, the idea of this work was born at a conference for his 60th birthday.

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I do not know if I would have ever arrived at the end of this paper without Salvatore, who had the difficult role of indicating me the

purpose of concluding this work as a priority. In particular during the drawing up of the paper, he had to fight hard against my resistance to cut the myriad of other “priorities” which absorb much of my concentration and time, and against my delaying attitude of, as Penelope, always undoing what I have done. I thank him for believing in the importance of my work in my and our life.

§1. PRELIMINARIES: DYNKIN DIAGRAMS.

For the preliminary material in this section see [Bo] and [K].

A Dynkin diagram Γ of finite or affine type is the datum (I, A) of its set of indices I and its Cartan matrix $A = (a_{ij})_{i,j \in I} \in \mathcal{M}_{n \times n}(\mathbb{Z})$ with the following properties:

- i) $a_{ii} = 2 \ \forall i \in I$;
- ii) $a_{ij} \leq 0 \ \forall i \neq j \in I$;
- iii) $a_{ij} = 0 \Leftrightarrow a_{ji} = 0$;
- iv) the determinants of all the proper principal minors of A are positive, and $\det(A) \geq 0$ (Γ is of finite type if $\det(A) > 0$ and of affine type if $\det(A) = 0$);

Γ is said to be indecomposable if furthermore:

- v) if $I = I' \cup I''$ with $I' \cap I'' = \emptyset$ and $I', I'' \neq \emptyset$ then $\exists i' \in I', i'' \in I''$ such that $a_{i'i''} \neq 0$.

Between the vertices $i \neq j \in I$ there are $\max\{|a_{ij}|, |a_{ji}|\}$ edges, with an arrow pointing at i if $|a_{ij}| > |a_{ji}|$; vertices, edges and arrows uniquely determine Γ .

A Dynkin diagram automorphism of Γ is a map $\chi: I \rightarrow I$ such that $a_{\chi(i)\chi(j)} = a_{ij} \ \forall i, j \in I$.

It is universally known that these data are classified (see [Bo]); the type of the indecomposable finite data is denoted by $X_{\#I}$ ($X = A, B, C, D, E, F, G$).

In this preliminary section we recall the construction and classification of the indecomposable Dynkin diagrams of affine type due to Kac (see [K]) and fix the general notations used in the paper.

Let $\tilde{\Gamma}$ be an indecomposable Dynkin diagram of finite type, with set of vertices \tilde{I} ($\#\tilde{I} = \tilde{n}$) and Cartan matrix $\tilde{A} = (\tilde{a}_{i'j'})_{i',j' \in \tilde{I}}$. To $X_{\tilde{n}}$ it is attached:

- a) the root lattice $\tilde{Q} = \bigoplus_{i' \in \tilde{I}} \mathbb{Z} \tilde{\alpha}_{i'}$;
- b) the Weyl group $\tilde{W} \subseteq \text{Aut}(\tilde{Q})$ generated by the reflections $\{\tilde{s}_{i'} | i' \in \tilde{I}\}$ where $\tilde{s}_{i'}$ is defined by $\tilde{s}_{i'}(\tilde{\alpha}_{j'}) = \tilde{\alpha}_{j'} - \tilde{a}_{i'j'} \tilde{\alpha}_{i'} \ (i', j' \in \tilde{I})$;

c) the (uniquely determined up to a scalar factor) \tilde{W} -invariant bilinear form $(\cdot|\cdot)$ on \tilde{Q} , which induces a positive definite scalar product on $\mathbb{R} \otimes_{\mathbb{Z}} \tilde{Q} = \oplus_{i' \in \tilde{I}} \mathbb{R} \tilde{\alpha}_{i'}$;

d) the root system $\tilde{\Phi} \subseteq \tilde{Q}$, which is the \tilde{W} -orbit of the set $\{\tilde{\alpha}_{i'} | i' \in \tilde{I}\}$ and is also characterized by the property $\tilde{\Phi} = \{\tilde{\alpha} \in \tilde{Q} | \exists i' \in \tilde{I} \text{ such that } (\tilde{\alpha}|\tilde{\alpha}) = (\tilde{\alpha}_{i'}|\tilde{\alpha}_{i'})\}$.

A Dynkin diagram automorphism χ induces an orthogonal transformation χ of $(\tilde{Q}, (\cdot|\cdot))$ ($\chi(\tilde{\alpha}_{i'}) = \tilde{\alpha}_{\chi(i')}$), and we have that $\chi \circ \tilde{s}_{i'} = \tilde{s}_{\chi(i')} \circ \chi$, $\chi(\tilde{\Phi}) = \tilde{\Phi}$.

Consider the datum $(X_{\tilde{n}}, \chi)$, with χ Dynkin diagram automorphism of $X_{\tilde{n}}$ and let k be the order of χ . It is well known (see [K]) that to this datum it is possible to attach an indecomposable Dynkin diagram of affine type Γ and an indecomposable subdiagram of finite type $\Gamma_0 \hookrightarrow \Gamma$ with the following properties:

I) the set of vertices I of Γ and I_0 of Γ_0 are $I_0 = \tilde{I}/\chi$ (the set of χ -orbits in \tilde{I} : $\forall i' \in \tilde{I}$ denote by $\bar{i'} \in I_0$ the χ -orbit of i') and $I = I_0 \cup \{0\}$; we shall denote by n the cardinality of I_0 and by $\{1, \dots, n\}$ the set I_0 (so that $I = \{0, 1, \dots, n\}$);

II) the Cartan matrix A_0 of Γ_0 is connected to \tilde{A} through the relation

$$a_{\bar{i'} \bar{j'}} = 2 \frac{\sum_{u \in \mathbb{Z}/k\mathbb{Z}} \tilde{a}_{\chi^u(i')j'}}{\sum_{u \in \mathbb{Z}/k\mathbb{Z}} \tilde{a}_{\chi^u(i')i'}};$$

remark in particular that if $k = 1$ we have $I_0 = \tilde{I}$ and $A_0 = \tilde{A}$, hence $\Gamma_0 = \tilde{\Gamma}$;

III) the root lattice $Q_0 = \oplus_{i \in I_0} \mathbb{Z} \alpha_i$ of Γ_0 naturally embeds in the root lattice $Q = \oplus_{i \in I} \mathbb{Z} \alpha_i$ of Γ ; their positive subsets are $Q_{0,+} = \sum_{i \in I_0} \mathbb{N} \alpha_i$ and $Q_+ = \sum_{i \in I} \mathbb{N} \alpha_i$;

IV) the highest root ϑ_0 of Γ_0 is characterized by the properties that $\vartheta_0 \in \Phi_0$ (the root system of Γ_0) and $\vartheta_0 - \alpha \in Q_{0,+} \forall \alpha \in \Phi_0$; it has also the property that $(\vartheta_0|\vartheta_0) \geq (\alpha|\alpha) \forall \alpha \in \Phi_0$;

V) the highest shortest root $\vartheta_0^{(s)}$ of Γ_0 is characterized by the properties that $\vartheta_0^{(s)} \in \Phi_0$, $(\vartheta_0^{(s)}|\vartheta_0^{(s)}) \leq (\alpha|\alpha) \forall \alpha \in \Phi_0$ and $\vartheta_0^{(s)} - \alpha \in Q_{0,+} \forall \alpha \in \Phi_0$ such that $(\alpha|\alpha) = (\vartheta_0^{(s)}|\vartheta_0^{(s)})$;


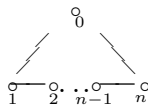
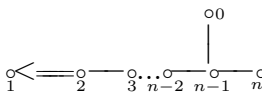
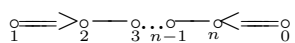
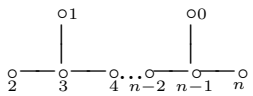
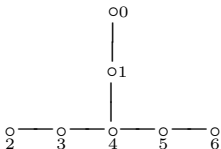
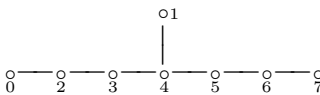
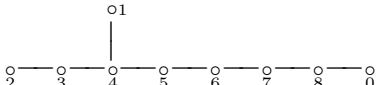
VI) the Cartan matrix A of Γ extends A_0 : $A = (a_{ij})_{i,j \in I}$, with

$$a_{00} = 2 \text{ and, } \forall i \in I_0, \ a_{0i} = -2 \frac{(\theta|\alpha_i)}{(\theta|\theta)}, \ a_{i0} = -2 \frac{(\alpha_i|\theta)}{(\alpha_i|\alpha_i)},$$

$$\text{where } \theta = \begin{cases} \vartheta_0 & \text{if } k = 1 \\ 2\vartheta_0^{(s)} & \text{if } X_{\tilde{n}} = A_{2n} \text{ and } \chi \neq id \\ \vartheta_0^{(s)} & \text{otherwise.} \end{cases}$$

The type of the Dynkin diagram Γ thus constructed is denoted by $X_n^{(k)}$ (indeed it does not depend on χ but just on k), and it is well known (see [K]) that this construction provides a classification of the indecomposable affine Dynkin diagrams, that we list in the following table.

The labels under the vertices fix an identification between I and $\{0, 1, \dots, n\}$ such that I_0 corresponds to $\{1, \dots, n\}$. For each type we also recall the coefficients r_i (for $i \in I_0$) in the expression $\theta = \sum_{i \in I_0} r_i \alpha_i$ (remark that we correct here a missprint in [Da]: the coefficient r_n for case $A_{2n-1}^{(2)}$).

$X_{\tilde{n}=\tilde{n}(n)}^{(k)}$	n	(Γ, I)	(r_1, \dots, r_n)
$A_1^{(1)}$	1		(1)
$A_n^{(1)}$	> 1		$(1, \dots, 1)$
$B_n^{(1)}$	> 2		$(2, \dots, 2, 1)$
$C_n^{(1)}$	> 1		$(1, 2, \dots, 2)$
$D_n^{(1)}$	> 3		$(1, 1, 2, \dots, 2, 1)$
$E_6^{(1)}$	6		$(2, 1, 2, 3, 2, 1)$
$E_7^{(1)}$	7		$(2, 2, 3, 4, 3, 2, 1)$
$E_8^{(1)}$	8		$(3, 2, 4, 6, 5, 4, 3, 2)$

$F_4^{(1)}$	4	$\circ_1 - \circ_2 \leq \circ_3 - \circ_4 - \circ_0$	$(2, 4, 3, 2)$
$G_2^{(1)}$	2	$\circ_1 \leq \equiv \circ_2 - \circ_0$	$(3, 2)$
$A_2^{(2)}$	1	$\circ_1 \leq \equiv \circ_0$	(2)
$A_{2n}^{(2)}$	> 1	$\circ_1 \leq \circ_2 - \circ_3 \dots \circ_{n-1} \leq \circ_n - \circ_0$	$(2, \dots, 2)$
$A_{2n-1}^{(2)}$	> 2	$\circ_1 \leq \circ_2 - \circ_3 \dots \circ_{n-2} - \circ_{n-1} \overset{\circ_0}{\overset{\circ_0}{\mid}} \circ_n$	$(1, 2, \dots, 2, 1)$
$D_{n+1}^{(2)}$	> 1	$\circ_1 \leq \circ_2 - \circ_3 \dots \circ_{n-1} \leq \circ_n \geq \circ_0$	$(1, \dots, 1)$
$E_6^{(2)}$	4	$\circ_0 - \circ_1 - \circ_2 \leq \circ_3 - \circ_4$	$(2, 3, 2, 1)$
$D_4^{(3)}$	2	$\circ_0 - \circ_1 \leq \equiv \circ_2$	$(2, 1)$

§2. PRELIMINARIES: WEYL GROUP and ROOT SYSTEM.

The following structures of the affine Weyl group and root system (see [Bo], [IM], [K], [M]) will be used in the paper:

i) the Weyl group $W_0 = \langle s_i | i \in I_0 \rangle \subseteq \text{Aut}(Q_0)$ of Γ_0 acts on Q by $s_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i \ \forall i \in I_0, j \in I$ and this action extends to the Weyl group $W = \langle s_i | i \in I \rangle \subseteq \text{Aut}(Q)$ of Γ by $s_0(\alpha_i) = \alpha_i - a_{0i}\alpha_0 \ \forall i \in I$;

ii) the W -invariant bilinear form $(\cdot | \cdot)$ on Q induces a positive semidefinite symmetric bilinear form on $\mathbb{R} \otimes_{\mathbb{Z}} Q$: it is obviously positive definite on $\mathbb{R} \otimes_{\mathbb{Z}} Q_0$, and has kernel generated by $\delta = \alpha_0 + \theta = \sum_{i \in I} r_i \alpha_i \in Q$ where $r_0 = 1$ always;

iii) $(\cdot | \cdot)$ can be uniquely normalized in such a way that there is a diagonal matrix $D = \text{diag}(d_i | i \in I)$ with $1 \in \{d_i | i \in I_0\} \subseteq \{d_i | i \in I\} \subseteq \mathbb{Z}_+$ and $(\alpha_i | \alpha_j) = d_i a_{ij} \ \forall i, j \in I; \ \forall i \in I, w \in W$ set $d_{w(\alpha_i)} = d_i$;

iv) $\forall i \in I_0$ define $\tilde{d}_i = \begin{cases} 1 & \text{if } k = 1 \text{ or } X_{\tilde{n}}^{(k)} = A_{2n}^{(2)} \\ d_i & \text{otherwise;} \end{cases}$

v) the weight lattice $\hat{P} \subseteq \mathbb{R} \otimes_{\mathbb{Z}} Q_0$ is $\hat{P} = \oplus_{i \in I_0} \mathbb{Z} \lambda_i$, where $\forall i \in I_0$ $\lambda_i \in \mathbb{R} \otimes_{\mathbb{Z}} Q_0$ is defined by $(\lambda_i | \alpha_j) = \tilde{d}_i \delta_{ij} \ \forall j \in I_0$; Q_0 naturally embeds in \hat{P} , which provides a W -invariant action on Q by $x(\alpha) = \alpha - (x | \alpha) \delta \ \forall x \in \hat{P}, \ \alpha \in Q$;

vi) as subgroups of $Aut(Q)$ we have $W \leq \hat{P} \rtimes W_0$; $\hat{W} = \hat{P} \rtimes W_0$ is called the extended Weyl group of Γ and we have also $\hat{W} = W \rtimes \mathcal{T}$, where $\mathcal{T} = Aut(\Gamma) \cap \hat{W}$;

vii) the extended braid group $\hat{\mathcal{B}}$ is the group generated by $\{T_w | w \in \hat{W}\}$ with relations $T_w T_{w'} = T_{ww'}$ whenever $l(ww') = l(w)l(w')$, where $l : \hat{W} \rightarrow \mathbb{N}$ is defined by

$$l(w) = \min\{r \in \mathbb{N} | \exists i_1, \dots, i_r \in I, \tau \in \mathcal{T} \text{ such that } w = s_{i_1} \cdot \dots \cdot s_{i_r} \tau\};$$

set $T_i = T_{s_i} \forall i \in I$; recall that $l(\sum_{i \in I_0} m_i \lambda_i) = \sum_{i \in I_0} m_i l(\lambda_i)$ if $m_i \in \mathbb{N} \forall i \in I_0$;

viii) the root system Φ of Γ decomposes into the union of the sets Φ^{re} of the real roots and Φ^{im} of the imaginary roots, where Φ^{re} is the W -orbit in Q of the set $\{\alpha_i | i \in I\}$ and $\Phi^{\text{im}} = \{m\delta | m \in \mathbb{Z} \setminus \{0\}\}$; the set of positive roots is $\Phi_+ = \Phi \cap Q_+$;

ix) the multiplicity of the root $\alpha \in \Phi$ is 1 if α is real and $\#\{i \in I_0 | \tilde{d}_i | m\}$ if $\alpha = m\delta$ ($m \in \mathbb{Z} \setminus \{0\}$); the set $\hat{\Phi}$ of roots with multiplicity is $\hat{\Phi} = \Phi^{\text{re}} \cup \hat{\Phi}^{\text{im}}$ where $\hat{\Phi}^{\text{im}} = \{(m\delta, i) | i \in I_0, m \in \mathbb{Z} \setminus \{0\}, \tilde{d}_i | m\}$; the set of positive roots with multiplicities is $\hat{\Phi}_+ = \Phi_+^{\text{re}} \cup \hat{\Phi}_+^{\text{im}} = (\Phi_+ \cap \Phi^{\text{re}}) \cup \{(m\delta, i) \in \hat{\Phi} | m > 0\}$;

x) choose a sequence $\iota : \mathbb{Z} \ni r \mapsto \iota_r \in I$ such that $\forall i \in I_0$ $s_{\iota_1} \cdot \dots \cdot s_{\iota_{N_i}} \tau_i = \sum_{j=1}^i \lambda_j$ and $\forall r \in \mathbb{Z}$ $\iota_{r+N_n} = \tau_n(\iota_r)$, where $N_i = \sum_{j=1}^i l(\lambda_j)$ and $\tau_i \in \mathcal{T}$; then ι induces a map

$$\mathbb{Z} \ni r \mapsto w_r \in W \text{ defined by } w_r = \begin{cases} s_{\iota_1} \cdot \dots \cdot s_{\iota_{r-1}} & \text{if } r \geq 1 \\ s_{\iota_0} \cdot \dots \cdot s_{\iota_{r+1}} & \text{if } r \leq 0 \end{cases}$$

and a bijection

$$\mathbb{Z} \ni r \mapsto \beta_r = w_r(\alpha_{\iota_r}) \in \Phi_+^{\text{re}};$$

xi) the total ordering \preceq of $\hat{\Phi}_+$ defined by

$$\beta_r \preceq \beta_{r-1} \preceq (\tilde{m}\delta, i) \preceq (m\delta, j) \preceq (m\delta, i) \preceq \beta_{s+1} \preceq \beta_s$$

$$\forall r \leq 0, s \geq 1, \tilde{m} > m > 0, j \leq i \in I_0$$

induces on Φ_+ a convex ordering: if $\alpha = \sum_{r=1}^M \gamma_r$ with $M > 1$, $\gamma_1 \preceq \dots \preceq \gamma_M$ and $\alpha, \gamma_r \in \Phi_+ \forall r = 1, \dots, M$, then either $\gamma_1 \prec \alpha$ or $\gamma_r \in \Phi^{\text{im}} \forall r = 1, \dots, M$.

§3. PRELIMINARIES: the DRINFELD-JIMBO PRESENTATION \mathcal{U}_q .

In this section we recall the definition of the quantum algebra \mathcal{U}_q introduced by Drinfeld and Jimbo (see [Dr2] and [Jm]), and the structures and results (see [Be], [Da], [LS],[L]) needed in §12. First of all recall some notations.

Notation 1.

- i) For all $i \in I_0$ we denote by q_i the element $q_i = q^{d_i} \in \mathbb{C}(q)$.
- ii) Consider the ring $\mathbb{Z}[x, x^{-1}]$. Then for all $m, r \in \mathbb{Z}$ the elements $[m]_x$, $[m]_x!$ ($m \geq 0$) and $\begin{bmatrix} m \\ r \end{bmatrix}_x$ ($m \geq r \geq 0$) are defined respectively by $[m]_x = \frac{x^m - x^{-m}}{x - x^{-1}}$, $[m]_x! = \prod_{s=1}^m [s]_x$ and $\begin{bmatrix} m \\ r \end{bmatrix}_x = \frac{[m]_x!}{[r]_x! [m-r]_x!}$, which all lie in $\mathbb{Z}[x, x^{-1}]$.
- iii) Consider the field $\mathbb{C}(q)$ and, given $v \in \mathbb{C}(q) \setminus \{0\}$, the natural homomorphism $\mathbb{Z}[x, x^{-1}] \rightarrow \mathbb{C}(q)$ determined by the condition $x \mapsto v$; then for all $m, r \in \mathbb{Z}$ the elements $[m]_v$, $[m]_v!$ ($m \geq 0$) and $\begin{bmatrix} m \\ r \end{bmatrix}_v$ ($m \geq r \geq 0$) denote the images in $\mathbb{C}(q)$ respectively of the elements $[m]_x$, $[m]_x!$ and $\begin{bmatrix} m \\ r \end{bmatrix}_x$.

Definition 2.

Let $\Gamma = (I, A)$ be a Dynkin diagram of finite or affine type.

- i) The (Drinfeld-Jimbo) quantum algebra of type Γ is the $\mathbb{C}(q)$ -algebra $\mathcal{U}_q = \mathcal{U}_q(\Gamma)$ generated by

$$\{E_i, F_i, K_i^{\pm 1} | i \in I\}$$

with relations:

$$K_i K_i^{-1} = 1 = K_i^{-1} K_i, \quad K_i K_j = K_j K_i \quad \forall i, j \in I,$$

$$K_i E_j = q_i^{a_{ij}} E_j K_i, \quad K_i F_j = q_i^{-a_{ij}} F_j K_i \quad \forall i, j \in I,$$

$$[E_i, F_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}} \quad \forall i, j \in I,$$

$$\sum_{u=0}^{1-a_{ij}} \begin{bmatrix} 1-a_{ij} \\ u \end{bmatrix}_{q_i} E_i^u E_j E_i^{1-a_{ij}-u} = 0 \quad \forall i \neq j \in I,$$

$$\sum_{u=0}^{1-a_{ij}} \begin{bmatrix} 1-a_{ij} \\ u \end{bmatrix}_{q_i} F_i^u F_j F_i^{1-a_{ij}-u} = 0 \quad \forall i \neq j \in I;$$

the last two sets of relations are called the Serre relations.

If Γ is affine of type $X_{\tilde{n}}^{(k)}$ we also set:

- ii) $\mathcal{U}_q^{DJ} = \mathcal{U}_q^{DJ}(X_{\tilde{n}}^{(k)}) = \mathcal{U}_q(\Gamma)$, to stress the distinction of this affine quantum algebra from its Drinfeld realization;

- iii) $\mathcal{U}_q^{fin} = \mathcal{U}_q^{fin}(X_{\tilde{n}}^{(k)}) = \mathcal{U}_q(\Gamma_0)$ (see section §1,I).

Definition 3.

Recall that \mathcal{U}_q is endowed with the following structures:

i) the Q -gradation $\mathcal{U}_q = \bigoplus_{\alpha \in Q} \mathcal{U}_{q,\alpha}$ determined by the conditions:

$$E_i \in \mathcal{U}_{q,\alpha_i}, \quad F_i \in \mathcal{U}_{q,-\alpha_i}, \quad K_i^{\pm 1} \in \mathcal{U}_{q,0} \quad \forall i \in I; \quad \mathcal{U}_{q,\alpha} \mathcal{U}_{q,\beta} \subseteq \mathcal{U}_{q,\alpha+\beta} \quad \forall \alpha, \beta \in Q;$$

ii) the triangular decomposition: $\mathcal{U}_q \cong \mathcal{U}_q^- \otimes \mathcal{U}_q^0 \otimes \mathcal{U}_q^+$, where \mathcal{U}_q^- , \mathcal{U}_q^0 and \mathcal{U}_q^+ are the subalgebras of \mathcal{U}_q generated respectively by $\{E_i | i \in I\}$, $\{K_i^{\pm 1} | i \in I\}$ and $\{F_i | i \in I\}$; in particular

$$\mathcal{U}_{q,\alpha} \cong \bigoplus_{\substack{\beta, \gamma \in Q_+ : \\ \gamma - \beta = \alpha}} \mathcal{U}_{q,-\beta}^- \otimes \mathcal{U}_q^0 \otimes \mathcal{U}_{q,\gamma}^+ \quad \forall \alpha \in Q$$

where $\mathcal{U}_{q,\alpha}^\pm = \mathcal{U}_{q,\alpha} \cap \mathcal{U}_q^\pm \quad \forall \alpha \in Q$;

iii) the \mathbb{C} -anti-linear anti-involution $\Omega : \mathcal{U}_q \rightarrow \mathcal{U}_q$ defined by

$$\Omega(q) = q^{-1}; \quad \Omega(E_i) = F_i, \quad \Omega(F_i) = E_i, \quad \Omega(K_i) = K_i^{-1} \quad \forall i \in I;$$

iv) the $\mathbb{C}(q)$ -linear anti-involution $\Xi : \mathcal{U}_q \rightarrow \mathcal{U}_q$ defined by

$$\Xi(E_i) = E_i, \quad \Xi(F_i) = F_i, \quad \Xi(K_i) = K_i^{-1} \quad \forall i \in I,$$

v) the braid group action defined by

$$T_i(K_j) = K_j K_i^{-a_{ij}} \quad \forall i, j \in I,$$

$$T_i(E_i) = -F_i K_i, \quad T_i(F_i) = -K_i^{-1} E_i \quad \forall i \in I,$$

$$T_i(E_j) = \sum_{r=0}^{-a_{ij}} (-1)^{r-a_{ij}} q_i^{-r} E_i^{(-a_{ij}-r)} E_j E_i^{(r)}, \quad T_i(F_j) = \Omega(T_i(E_j)) \quad \forall i \neq j \in I$$

where $\forall m \in \mathbb{N} \quad E_i^{(m)} = \frac{E_i^m}{[m]_{q_i}!}$;

vi) a natural $Aut(\Gamma)$ -action: $\tau(K_i) = K_{\tau(i)}$, $\tau(E_i) = E_{\tau(i)}$, $\tau(F_i) = F_{\tau(i)}$ for all $\tau \in Aut(\Gamma)$, $i \in I_0$; if Γ is affine then setting $T_\tau = \tau$ extends the braid group action to an extended braid group action;

vii) if $\Gamma \hookrightarrow \Gamma'$ is a Dynkin diagram embedding then the \mathbb{C} -homomorphism

$$\varphi_{\Gamma, \Gamma'} : \mathcal{U}_q(\Gamma) \rightarrow \mathcal{U}_q(\Gamma')$$

is naturally defined by

$$q \mapsto q^{\min\{d'_i | i \in I\}}, \quad K_i^{\pm 1} \mapsto K_i^{\pm 1}, \quad E_i \mapsto E_i, \quad F_i \mapsto F_i \quad (i \in I);$$

in particular if Γ is of affine type $\varphi = \varphi_{\Gamma_0, \Gamma} : \mathcal{U}_q^{fin} \rightarrow \mathcal{U}_q^{DJ}$ is a $\mathbb{C}(q)$ -homomorphism;

viii) positive and negative root vectors $E_\alpha \in \mathcal{U}_{q,\alpha}^{DJ,+}$ and $F_\alpha = \Omega(E_\alpha) \in \mathcal{U}_{q,-\alpha}^{DJ,-}$ ($\alpha \in \hat{\Phi}_+$) such that if Γ is of affine type $E_{\beta_r} = T_{w_r}(E_{\iota_r})$ if $r \geq 1$, $E_{\beta_r} = T_{w_r}^{-1}(E_{\iota_r})$ if $r \leq 0$, and $E_{(\tilde{d}_i r \delta, i)} = -E_{\tilde{d}_i r \delta - \alpha_i} E_i + q_i^{-2} E_i E_{\tilde{d}_i r \delta - \alpha_i}$ if $r > 0$, $i \in I_0$.

Remark 4.

We have that:

- i) $\Omega\Xi = \Xi\Omega$, $\Omega T_i = T_i\Omega \forall i \in I$ and $\Omega\tau = \tau\Omega \forall \tau \in \mathcal{T}$;
- ii) $\Xi T_i = T_i^{-1}\Xi \forall i \in I$ and $\Xi\tau = \tau\Xi \forall \tau \in \mathcal{T}$;
- moreover if Γ is of affine type:
- iii) φ commutes with Ω , Ξ and T_i ($i \in I_0$).
- iv) in cases $A_1^{(1)}$ and $A_2^{(2)}$ $\Xi T_1 T_{\lambda_1} = T_{\lambda_1}^{-1} \Xi T_1$ (recall that $T_{\lambda_1} = T_0 T_\tau = T_\tau T_1$, where $\langle \tau \rangle = \text{Aut}(\Gamma)$, in case $A_1^{(1)}$ and $T_{\lambda_1} = T_0 T_1$ in case $A_2^{(2)}$);
- v) $T_w(\mathcal{U}_{q,\alpha}^{DJ}) = \mathcal{U}_{q,w(\alpha)}^{DJ} \forall w \in \hat{W}$, $\alpha \in Q$;
- vi) $T_w(E_i) \in \mathcal{U}_{q,w(\alpha_i)}^{DJ,+}$ if $w \in \hat{W}$ and $i \in I$ are such that $w(\alpha_i) \in Q_+$ (i.e. $l(ws_i) > l(w)$);
- vii) $E_{m\bar{d}_i\delta+\alpha_i} = T_{\lambda_i}^{-m}(E_i)$, $F_{m\bar{d}_i\delta+\alpha_i} = T_{\lambda_i}^{-m}(F_i) \forall m \in \mathbb{N}, i \in I_0$;
- viii) $\{K_\alpha | \alpha \in Q\}$ is a basis of $\mathcal{U}_q^{DJ,0}$, where $K_\alpha = \prod_{i \in I} K_i^{m_i}$ if $\alpha = \sum_{i \in I} m_i \alpha_i \in Q$;
- ix) $\{E(\gamma) = E_{\gamma_1} \cdots E_{\gamma_M} | M \in \mathbb{N}, \gamma = (\gamma_1 \preceq \dots \preceq \gamma_M), \gamma_h \in \hat{\Phi}_+ \forall h = 1, \dots, M\}$ is a basis of $\mathcal{U}_q^{DJ,+}$;
- x) $\{E(\gamma)K_\alpha\Omega(E(\gamma')) | \alpha \in Q, \gamma = (\gamma_1 \preceq \dots \preceq \gamma_M) \in \hat{\Phi}_+^M, \gamma' = (\gamma'_1 \preceq \dots \preceq \gamma'_{M'}) \in \hat{\Phi}_+^{M'}, M, M' \in \mathbb{N}\}$ is a basis of \mathcal{U}_q^{DJ} , called the PBW-basis;
- xi) $\forall \alpha \prec \beta \in \hat{\Phi}_+ E_\beta E_\alpha - q^{(\alpha|\beta)} E_\alpha E_\beta$ is a linear combination of $\{E(\gamma) | \gamma = (\gamma_1 \preceq \dots \preceq \gamma_M) \in \hat{\Phi}_+^M, M \in \mathbb{N}, \alpha \prec \gamma_1\}$ (Levendorskii-Soibelman formula).

Remark 5.

If Γ is affine remark 4,ix) implies that $\dim \mathcal{U}_{q,\alpha}^{DJ,+} = \dim \mathcal{U}_{q,\alpha}^{fin,+} \forall \alpha \in Q_{0,+}$. In particular φ is injective.

Definition 6.

If Γ is affine, for $i \in I_0$ let

$$\varphi_i : \begin{cases} \mathcal{U}_q^{DJ}(A_1^{(1)}) \rightarrow \mathcal{U}_q^{DJ}(X_{\tilde{n}}^{(k)}) & \text{if } (X_{\tilde{n}}^{(k)}, i) \neq (A_{2n}^{(2)}, 1) \\ \mathcal{U}_q^{DJ}(A_2^{(2)}) \rightarrow \mathcal{U}_q^{DJ}(X_{\tilde{n}}^{(k)}) & \text{if } (X_{\tilde{n}}^{(k)}, i) = (A_{2n}^{(2)}, 1) \end{cases}$$

be the \mathbb{C} -homomorphisms defined on the generators as follows:

$$q \mapsto q_i, K_1^{\pm 1} \mapsto K_i^{\pm 1}, E_1 \mapsto E_i, F_1 \mapsto F_i$$

and

$$\begin{aligned} K_0 &\mapsto K_{\bar{d}_i\delta-\alpha_i}, E_0 \mapsto E_{\bar{d}_i\delta-\alpha_i}, F_0 \mapsto F_{\bar{d}_i\delta-\alpha_i} & \text{if } (X_{\tilde{n}}^{(k)}, i) \neq (A_{2n}^{(2)}, 1) \\ K_0 &\mapsto K_{\delta-2\alpha_1}, E_0 \mapsto E_{\delta-2\alpha_1}, F_0 \mapsto F_{\delta-2\alpha_1} & \text{if } (X_{\tilde{n}}^{(k)}, i) = (A_{2n}^{(2)}, 1). \end{aligned}$$

Remark 7.

- i) $\varphi_i\Omega = \Omega\varphi_i$, $\varphi_i T_1 = T_i\varphi_i$ and $\varphi_i T_{\lambda_1} = T_{\lambda_i}\varphi_i \forall i \in I_0$;

ii) φ_i ($i \in I_0$) is injective (thanks to the PBW-bases).

§4. The DRINFELD REALIZATION \mathcal{U}_q^{Dr} : DEFINITION.

In this section the definition of the Drinfeld realization $\mathcal{U}_q^{Dr}(X_{\tilde{n}}^{(k)})$ of the affine quantum algebra of type $X_{\tilde{n}}^{(k)}$ is presented; the definition is discussed and reformulated using the set $I_0 \times \mathbb{Z}$ as index set for the generators instead of the set $\tilde{I} \times \mathbb{Z}$ used in [Dr1] and followed in literature (see for instance [CP3], [Jn], [JZ2]), because the relations translated from $\tilde{I} \times \mathbb{Z}$ to $I_0 \times \mathbb{Z}$ seem simpler to handle, even though they lose the immediate connection with the datum (\tilde{I}, χ) . This reformulation, which is useful if one aims to compare the Drinfeld realization with the Drinfeld-Jimbo presentation, is not difficult, but it is presented with some care in order to avoid any ambiguity.

Notation 1.

- i) ω denotes a primitive k^{th} root of 1.
- ii) Fix the normalization of the \tilde{W} -invariant bilinear form $(\cdot|\cdot)$ on \tilde{Q} such that $\min\{\sum_{u \in \mathbb{Z}/k\mathbb{Z}} (\tilde{\alpha}_{i'}|\tilde{\alpha}_{\chi^u(i')})|i' \in \tilde{I}\} = 2$.
- iii) Denote by \tilde{d} the number $\tilde{d} = \max\{\tilde{d}_i|i \in I_0\}$ (in case $A_{2n}^{(2)}$ $\tilde{d} = 1$, otherwise $\tilde{d} = k$).
- iv) Let Y be a function from \mathbb{Z}^l ($l \in \mathbb{N}$) to any algebra; given $\sigma \in \mathcal{S}_l$ and $p = (p_1, \dots, p_l) \in \mathbb{Z}^l$ set $\sigma.(Y(p)) = Y(\sigma.p) = Y(p_{\sigma^{-1}(1)}, \dots, p_{\sigma^{-1}(l)})$.
- v) Analogously if $f \in \mathbb{C}(q)[[u_1^{\pm 1}, \dots, u_l^{\pm 1}]]$ and $u = (u_1, \dots, u_l)$ define $\sigma.(f(u))$ by $\sigma.(f(u)) = f(u_{\sigma^{-1}(1)}, \dots, u_{\sigma^{-1}(l)})$ for all $\sigma \in \mathcal{S}_l$.
- vi) By “ (R^{\pm}) is the relation $S^{\pm} = 0$ ” it is meant “ (R^+) is the relation $S^+ = 0$ and (R^-) is the relation $S^- = 0$ ”.
- vii) More generally “ A^{\pm} has the property P^{\pm} ” means “ A^+ has the property P^+ and A^- has the property P^- ”.

For the definition of the Drinfeld realization of affine quantum algebras, that we recall here, see [Dr1].

Definition 2.

Let $X_{\tilde{n}}^{(k)}$ be a Dynkin diagram of affine type; the Drinfeld realization of the quantum algebra of type $X_{\tilde{n}}^{(k)}$ is the $\mathbb{C}(q)$ -algebra $\mathcal{U}_q^{Dr}(X_{\tilde{n}}^{(k)}) = \mathcal{U}_q^{Dr}$ generated by

$$(\mathcal{G}) \quad \mathcal{C}^{\pm 1}, \quad \mathcal{K}_{i'}^{\pm 1} \quad (i' \in \tilde{I}), \quad \mathcal{X}_{i',r}^{\pm} \quad ((i', r) \in \tilde{I} \times \mathbb{Z}), \quad \mathcal{H}_{i',r} \quad ((i', r) \in \tilde{I} \times (\mathbb{Z} \setminus \{0\})),$$

with the following relations (\mathcal{DR}):

$$(\mathcal{Z}) \quad \mathcal{K}_{\chi(i')} = \mathcal{K}_{i'}, \quad \mathcal{H}_{\chi(i'),r} = \omega^r \mathcal{H}_{i',r} \quad (i' \in \tilde{I}, \quad r \in \mathbb{Z} \setminus \{0\}),$$

$$(\mathcal{Z}\mathcal{X}^\pm) \quad \mathcal{X}_{\chi(i'),r}^\pm = \omega^r \mathcal{X}_{i',r}^\pm \quad ((i', r) \in \tilde{I} \times \mathbb{Z}),$$

$$(\mathcal{C}) \quad \mathcal{C}\mathcal{C}^{-1} = 1, \quad [\mathcal{C}, x] = 0 \quad \forall x,$$

$$(\mathcal{KK}) \quad \mathcal{K}_{i'}\mathcal{K}_{i'}^{-1} = 1 = \mathcal{K}_{i'}^{-1}\mathcal{K}_{i'}, \quad \mathcal{K}_{i'}\mathcal{K}_{j'} = \mathcal{K}_{j'}\mathcal{K}_{i'} \quad (i', j' \in \tilde{I}),$$

$$(\mathcal{K}\mathcal{X}^\pm) \quad \mathcal{K}_{i'}\mathcal{X}_{j',r}^\pm = q^{\pm \sum_{u \in \mathbb{Z}/k\mathbb{Z}} (\tilde{\alpha}_{i'} | \tilde{\alpha}_{\chi^u(j')})} \mathcal{X}_{j',r}^\pm \mathcal{K}_{i'} \quad (i', j' \in \tilde{I}, \quad r \in \mathbb{Z}),$$

$$(\mathcal{KH}) \quad [\mathcal{K}_{i'}, \mathcal{H}_{j',r}] = 0 \quad (i', j' \in \tilde{I}, \quad r \in \mathbb{Z} \setminus \{0\}),$$

$$(\mathcal{XX})$$

$$[\mathcal{X}_{i',r}^+, \mathcal{X}_{j',s}^-] = \frac{\sum_{u=0}^{k-1} \delta_{\chi^u(i'),j'} \omega^{us}}{\sum_{u=0}^{k-1} \delta_{\chi^u(i'),i'}} \cdot \frac{\mathcal{C}^{-s} \mathcal{K}_{i'} \tilde{\mathcal{H}}_{i',r+s}^+ - \mathcal{C}^{-r} \mathcal{K}_{i'}^{-1} \tilde{\mathcal{H}}_{i',r+s}^-}{(q - q^{-1}) [\frac{1}{2} \sum_{u \in \mathbb{Z}/k\mathbb{Z}} (\tilde{\alpha}_{\chi^u(i')} | \tilde{\alpha}_{i'})]_q}$$

$$((i', r), (j', s) \in \tilde{I} \times \mathbb{Z}),$$

$$(\mathcal{HX}^\pm)$$

$$[\mathcal{H}_{i',r}, \mathcal{X}_{j',s}^\pm] = \pm \tilde{b}_{i'j'r} \mathcal{C}^{\frac{r \mp |r|}{2}} \mathcal{X}_{j',r+s}^\pm \quad ((i', r) \in \tilde{I} \times (\mathbb{Z} \setminus \{0\}), (j', s) \in \tilde{I} \times \mathbb{Z}),$$

$$(\mathcal{HH}) \quad [\mathcal{H}_{i',r}, \mathcal{H}_{j',s}] = \delta_{r+s,0} \tilde{b}_{i'j'r} \frac{C^r - C^{-r}}{(q - q^{-1}) [\frac{1}{2} \sum_{u \in \mathbb{Z}/k\mathbb{Z}} (\tilde{\alpha}_{\chi^u(j')} | \tilde{\alpha}_{j'})]_q}$$

$$((i', r), (j', s) \in \tilde{I} \times (\mathbb{Z} \setminus \{0\})),$$

$$(\mathcal{XFG}^\pm)$$

$$F_{i'j'}^\pm(u_1, u_2) \mathcal{X}_{i'}^\pm(u_1) \mathcal{X}_{j'}^\pm(u_2) = G_{i'j'}^\pm(u_1, u_2) \mathcal{X}_{j'}^\pm(u_2) \mathcal{X}_{i'}^\pm(u_1) \quad (i', j' \in \tilde{I}),$$

$$(\mathcal{X3}^{\varepsilon,\pm})$$

$$\sum_{\sigma \in \mathcal{S}_3} \sigma. ((q^{-3\varepsilon} u_1^{\pm\varepsilon} - (q + q^{-1}) u_2^{\pm\varepsilon} + q^{3\varepsilon} u_3^{\pm\varepsilon}) \mathcal{X}_{i'}^\pm(u_1) \mathcal{X}_{i'}^\pm(u_2) \mathcal{X}_{i'}^\pm(u_3)) = 0$$

$$(i' \in \tilde{I}, \quad \tilde{a}_{\chi(i')i'} = -1),$$

$$(\mathcal{S}^\pm)$$

$$\sum_{\sigma \in \mathcal{S}_{1-a_{ij}}} \sigma. \sum_{u=0}^{1-a_{ij}} (-1)^u \begin{bmatrix} 1-a_{ij} \\ u \end{bmatrix}_{q_i} \mathcal{X}_{i',p_1}^\pm \cdots \mathcal{X}_{i',p_u}^\pm \mathcal{X}_{j',v}^\pm \mathcal{X}_{i',p_{u+1}}^\pm \cdots \mathcal{X}_{i',p_{1-a_{ij}}}^\pm = 0$$

$$(k = 1, \quad i', j' \in \tilde{I}, \quad i' \neq j'),$$

$$(\mathcal{XP}^\pm) \quad \sum_{\sigma \in \mathcal{S}_2} \sigma. \left(P_{i'j'}^\pm(u_1, u_2) (\mathcal{X}_{j'}^\pm(v) \mathcal{X}_{i'}^\pm(u_1) \mathcal{X}_{i'}^\pm(u_2) + \right. \\ \left. - [2]_q^{m_{i'j'}} \mathcal{X}_{i'}^\pm(u_1) \mathcal{X}_{j'}^\pm(v) \mathcal{X}_{i'}^\pm(u_2) + \mathcal{X}_{i'}^\pm(u_1) \mathcal{X}_{i'}^\pm(u_2) \mathcal{X}_{j'}^\pm(v) \right) = 0$$

$(k > 1, i', j' \in \tilde{I}, \chi(i') \neq j', \tilde{a}_{i'j'} < 0)$,

where $\tilde{\mathcal{H}}_{i',r}^\pm$, $\tilde{b}_{i'j'r}$, $\mathcal{X}_{i'}^\pm(u)$, $F_{i'j'}^\pm(u_1, u_2)$, $G_{i'j'}^\pm(u_1, u_2)$, ε , $P_{i'j'}^\pm(u_1, u_2)$ and $m_{i'j'}$ are defined as follows:

$$\sum_{r \in \mathbb{Z}} \tilde{\mathcal{H}}_{i', \pm r}^\pm u^r = \exp \left(\pm(q - q^{-1}) \left[\frac{1}{2} \sum_{u \in \mathbb{Z}/k\mathbb{Z}} (\tilde{\alpha}_{\chi^u(i')} | \tilde{\alpha}_{i'}) \right]_q \sum_{r > 0} \mathcal{H}_{i', \pm r} u^r \right);$$

$$\tilde{b}_{i'j'r} = \frac{\sum_{u=0}^{k-1} [r(\tilde{\alpha}_{i'} | \tilde{\alpha}_{\chi^u(j')})]_q \omega^{ru}}{r \left[\frac{1}{2} \sum_{u \in \mathbb{Z}/k\mathbb{Z}} (\tilde{\alpha}_{i'} | \tilde{\alpha}_{\chi^u(i')}) \right]_q};$$

$$\mathcal{X}_{i'}^\pm(u) = \sum_{r \in \mathbb{Z}} \mathcal{X}_{i',r}^\pm u^{-r};$$

$$F_{i'j'}^\pm(u_1, u_2) = \prod_{\substack{v \in \mathbb{Z}/k\mathbb{Z} \\ \tilde{a}_{i', \chi^v(j')} \neq 0}} (u_1 - \omega^v q^{\pm(\tilde{\alpha}_{i'} | \tilde{\alpha}_{\chi^v(j')})} u_2);$$

$$G_{i'j'}^\pm(u_1, u_2) = \prod_{\substack{v \in \mathbb{Z}/k\mathbb{Z} \\ \tilde{a}_{i', \chi^v(j')} \neq 0}} (q^{\pm(\tilde{\alpha}_{i'} | \tilde{\alpha}_{\chi^v(j')})} u_1 - \omega^v u_2);$$

$$\varepsilon = \pm 1;$$

$$P_{i'j'}^\pm(u_1, u_2) = \begin{cases} 1 & \text{if } \tilde{a}_{i', \chi(i')} = 0 \text{ and } \chi(j') \neq j', \text{ or } \chi(i') = i' \\ \frac{q^{\pm 2k} u_1^k - u_2^k}{q^{\pm 2} u_1 - u_2} & \text{otherwise;} \end{cases}$$

$$m_{i'j'} = \begin{cases} \frac{k}{d} \sum_{u \in \mathbb{Z}/k\mathbb{Z}} \delta_{i' \chi^u(i')} & \text{if } \tilde{a}_{i', \chi(i')} = 0 \text{ and } \chi(j') \neq j', \text{ or } \chi(i') = i' \\ k & \text{otherwise.} \end{cases}$$

Remark 3.

In [Dr1] not all the relations $(\mathcal{X}3^{\varepsilon, \pm})$ appear, but just the relations $(\mathcal{X}3^{1, +})$ and $(\mathcal{X}3^{-1, -})$; relations $(\mathcal{X}3^{-1, +})$ and $(\mathcal{X}3^{1, -})$ are introduced in [CP3] as consequences of relations $(\mathcal{Z}) - (\mathcal{X}\mathcal{F}\mathcal{G}^\pm)$, $(\mathcal{X}3^{1, +})$, $(\mathcal{X}3^{-1, -})$, (\mathcal{S}^\pm) , $(\mathcal{X}\mathcal{P}^\pm)$, since their use simplifies some calculations, making evident some symmetries (the stability of the relations under the anti-automorphism $\tilde{\Omega}$ and the automorphism $\tilde{\Theta}$). Here we use relations $(\mathcal{X}3^{\varepsilon, \pm})$ for the same reasons of simplification (see remarks §8.3 and §8.5), proving in proposition §10.1 the equivalence stated in [CP3].

Remark 4.

i) For all $r \in \mathbb{Z}$, the algebra generated by $\{\mathcal{Y}_{i'} | i' \in \tilde{I}\}$ with relations $\{\mathcal{Y}_{\chi(i')} = \omega^r \mathcal{Y}_{i'} | i' \in \tilde{I}\}$ is isomorphic to the algebra generated by $\{Y_i | i \in I_0\}$ with relations $\{Y_i = \omega^{r \# \{i' \in \tilde{I} : \tilde{i}' = i\}} Y_i | i \in I_0\}$, where a section $\tilde{\cdot} : I_0 \rightarrow \tilde{I}$ induces an isomorphism $Y_i \mapsto \mathcal{Y}_{\tilde{i}}$.

ii) Consider $i' \in \tilde{I}$ and let $i \in I_0 = \tilde{I}/\chi$ be the χ -orbit of i' . Notice that $\chi(i') = i' \Leftrightarrow k|\tilde{d}_i$; more precisely $\sum_{u \in \mathbb{Z}/k\mathbb{Z}} \delta_{i', \chi^u(i')} = \tilde{d}_i$ and $\tilde{d}_i \# \{i' \in \tilde{I} : \bar{i}' = i\} = k$.

iii) For all $r \in \mathbb{Z}$, the algebra generated by $\{Y_i | i \in I_0\}$ with relations $\{Y_i = \omega^{\frac{kr}{\tilde{d}_i}} Y_i | i \in I_0\}$ is trivially isomorphic to the algebra generated by $\{Y_i | i \in I_0\}$ with relations $\{Y_i = 0 | \tilde{d}_i \nmid r\}$, which is trivially isomorphic to the free algebra generated by $\{Y_i | i \in I_0 \text{ such that } \tilde{d}_i | r\}$.

iv) Hence, for all $r \in \mathbb{Z}$, the algebra generated by $\{\mathcal{Y}_{i'} | i' \in \tilde{I}\}$ with relations $\{\mathcal{Y}_{\chi(i')} = \omega^r \mathcal{Y}_{i'} | i' \in \tilde{I}\}$ is isomorphic to the algebra generated by $\{Y_i | i \in I_0\}$ with relations $\{Y_i = 0 | \tilde{d}_i \nmid r\}$, where a section $\tilde{\cdot} : I_0 \rightarrow \tilde{I}$ induces an isomorphism $Y_i \mapsto \mathcal{Y}_{\tilde{i}}$.

v) Finally, the algebra generated by $\{\mathcal{Y}_{i',r} | i' \in \tilde{I}, r \in \mathbb{Z}\}$ with relations $\{\mathcal{Y}_{\chi(i'),r} = \omega^r \mathcal{Y}_{i',r} | i' \in \tilde{I}, r \in \mathbb{Z}\}$ is isomorphic to the algebra generated by $\{Y_{i,r} | i \in I_0, r \in \mathbb{Z}\}$ with relations $\{Y_{i,r} = 0 | \tilde{d}_i \nmid r\}$, or equivalently to the free algebra generated by $\{Y_{i,r} | i \in I_0, r \in \mathbb{Z} \text{ such that } \tilde{d}_i | r\}$.

Notation 5.

Let us denote by $I_{\mathbb{Z}}$ the set $I_{\mathbb{Z}} = \{(i, r) \in I_0 \times \mathbb{Z} | \tilde{d}_i | r\}$.

Corollary 6.

i) \mathcal{U}_q^{Dr} is (isomorphic to) an algebra generated by
 (G)
 $C^{\pm 1}, \quad k_i^{\pm 1} \quad (i \in I_0), \quad X_{i,r}^{\pm} \quad ((i, r) \in I_0 \times \mathbb{Z}), \quad H_{i,r} \quad ((i, r) \in I_0 \times (\mathbb{Z} \setminus \{0\}));$
the relations

$$(ZX^{\pm}) \quad X_{i,r}^{\pm} = 0 \quad \forall (i, r) \in (I_0 \times \mathbb{Z}) \setminus I_{\mathbb{Z}}$$

and

$$(ZH) \quad H_{i,r} = 0 \quad \forall (i, r) \in (I_0 \times \mathbb{Z}) \setminus I_{\mathbb{Z}},$$

hold in \mathcal{U}_q^{Dr} .

ii) \mathcal{U}_q^{Dr} is generated by
 (G')
 $C^{\pm 1}, \quad k_i^{\pm 1} \quad (i \in I_0), \quad X_{i,r}^{\pm} \quad ((i, r) \in I_{\mathbb{Z}}), \quad H_{i,r} \quad ((i, r) \in I_{\mathbb{Z}} \setminus (I_0 \times \{0\})).$

Remark 7.

The relations $(\mathcal{Z}\mathcal{X}^{\pm})$ are equivalent to the condition $\mathcal{X}_{\chi(i')}^{\pm}(u) = \mathcal{X}_{i'}^{\pm}(\omega^{-1}u)$ for all $i' \in \tilde{I}$.

Notation 8.

Given $i, j \in I_0$ we set $\tilde{d}_{ij} = \max\{\tilde{d}_i, \tilde{d}_j\}$.

Remark 9.

- i) If $\tilde{\alpha}_{i'}$ ($i' \in \tilde{I}$) is a short root then $(\tilde{\alpha}_{i'} | \tilde{\alpha}_{i'}) = \frac{2k}{d}$.
- ii) $\forall i' \in \tilde{I}$ we have $\sum_{u \in \mathbb{Z}/k\mathbb{Z}} (\tilde{\alpha}_{i'} | \tilde{\alpha}_{\chi^u(i')}) = 2d_{\tilde{i}'}$.

Remark 10.

- i) Remark that there exists a section $\sim : I_0 \rightarrow \tilde{I}$ such that given $i, j \in I_0$ we have that $a_{ij} \neq 0 \Rightarrow \tilde{a}_{\tilde{i}\tilde{j}} \neq 0$ (of course it is always true that $\tilde{a}_{i'j'} \neq 0 \Rightarrow a_{\tilde{i}'\tilde{j}'} \neq 0$);
- ii) let \sim be a section as in i); then, if $k > 1$, $d_i a_{ij} = \max\{d_i, d_j\} \tilde{a}_{\tilde{i}\tilde{j}}$.

Remark 11.

- i) The relations (\mathcal{KK}) , (\mathcal{KX}^\pm) and (\mathcal{KH}) are compatible with the relations (\mathcal{Z}) and (\mathcal{ZX}^\pm) , in the sense that for all $i', j' \in \tilde{I}$, $r \in \mathbb{Z}$, $s \in \mathbb{Z} \setminus \{0\}$,

$$(\mathcal{KK})_{\chi(i'), j'} = (\mathcal{KK})_{i', j'} = (\mathcal{KK})_{i', \chi(j')},$$

$$(\mathcal{KX}^\pm)_{\chi(i'), j', r} = (\mathcal{KX}^\pm)_{i', j', r}, \quad (\mathcal{KX}^\pm)_{i', \chi(j'), r} = \omega^r (\mathcal{KX}^\pm)_{i', j', r}$$

and

$$(\mathcal{KH})_{\chi(i'), j', s} = (\mathcal{KH})_{i', j', s}, \quad (\mathcal{KH})_{i', \chi(j'), s} = \omega^s (\mathcal{KH})_{i', j', s};$$

- ii) if $i, j \in I_0$ are such that $\tilde{i}' = i$, $\tilde{j}' = j$, $(\mathcal{KX}^\pm)_{i', j'}$ is equivalent to

$$\mathcal{K}_{i'} \mathcal{X}_{j', r}^\pm = q_i^{\pm a_{ij}} \mathcal{X}_{j', r}^\pm \mathcal{K}_{i'}$$

(see §1.II), remark 9,ii) and notation §3.1,i)).

Remark 12.

- i) If we apply χ to the expression $\sum_{r>0} \mathcal{H}_{i', \pm r} u^r$ ($i' \in \tilde{I}$) we get (see (\mathcal{Z}))

$$\sum_{r>0} \mathcal{H}_{\chi(i'), \pm r} u^r = \sum_{r>0} \mathcal{H}_{i', \pm r} (\omega^{\pm 1} u)^r.$$

- ii) From i) and from definition 2 we get

$$\sum_{r \in \mathbb{Z}} \tilde{\mathcal{H}}_{\chi(i'), \pm r}^\pm u^r = \sum_{r \in \mathbb{Z}} \tilde{\mathcal{H}}_{i', \pm r}^\pm (\omega^{\pm 1} u)^r \quad \forall i' \in \tilde{I},$$

that is $\tilde{\mathcal{H}}_{\chi(i'), r}^\pm = \omega^r \tilde{\mathcal{H}}_{i', r}^\pm \quad \forall (i', r) \in \tilde{I} \times (\mathbb{Z} \setminus \{0\})$.

- iii) The relations (\mathcal{XX}) are compatible with relations (\mathcal{Z}) and (\mathcal{ZX}^\pm) :

$$(\mathcal{XX})_{\chi(i'), r; j', s} = \omega^r (\mathcal{XX})_{i', r; j', s}, \quad (\mathcal{XX})_{i', r; \chi(j'), s} = \omega^s (\mathcal{XX})_{i', r; j', s}.$$

iv) If \sim is as in remark 10 and $i, j \in I_0$, then $(\mathcal{X}\mathcal{X})_{\tilde{i}, r; \tilde{j}, s}$ is equivalent to

$$[\mathcal{X}_{i,r}^+, \mathcal{X}_{j,s}^-] = \begin{cases} \delta_{ij} \frac{C^{-s} \mathcal{K}_i \tilde{\mathcal{H}}_{i,r+s}^+ - C^{-r} \mathcal{K}_i^{-1} \tilde{\mathcal{H}}_{i,r+s}^-}{(q_i - q_i^{-1})} & \text{if } \tilde{d}_j | s \\ 0 & \text{otherwise} \end{cases}$$

(see remark 4,ii)).

Remark 13.

Let $i', j' \in \tilde{I}$, $r \in \mathbb{Z}$; then:

- i) $\tilde{b}_{i'j'r} = \frac{1}{r} \sum_{u=0}^{k-1} \left[\frac{r(\tilde{\alpha}_{i'} | \tilde{\alpha}_{\chi^u(j')})}{\tilde{d}_{i'}} \right]_{q_i} \omega^{ru}$;
- ii) $\tilde{b}_{\chi(i')j'r} = \omega^r \tilde{b}_{i'j'r}$ and $\tilde{b}_{i'\chi(j')r} = \omega^{-r} \tilde{b}_{i'j'r}$;
- iii) the relations $(\mathcal{H}\mathcal{X}^\pm)$ and $(\mathcal{H}\mathcal{H})$ are compatible with the relations (\mathcal{Z}) and $(\mathcal{Z}\mathcal{X}^\pm)$:

$$(\mathcal{H}\mathcal{X}^\pm)_{\chi(i'), r; j', s} = \omega^r (\mathcal{H}\mathcal{X}^\pm)_{i', r; j', s}, \quad (\mathcal{H}\mathcal{X}^\pm)_{i', r; \chi(j'), s} = \omega^s (\mathcal{H}\mathcal{X}^\pm)_{i', r; j', s}$$

and

$$(\mathcal{H}\mathcal{H})_{\chi(i'), r; j', s} = \omega^r (\mathcal{H}\mathcal{H})_{i', r; j', s}, \quad (\mathcal{H}\mathcal{H})_{i', r; \chi(j'), s} = \omega^s (\mathcal{H}\mathcal{H})_{i', r; j', s}.$$

Notation 14.

Let $i, j \in I_0$, $r \in \mathbb{Z}$; b_{ijr} denotes the element of $\mathbb{C}(q)$

$$b_{ijr} = \begin{cases} 0 & \text{if } \tilde{d}_{i,j} \nmid r \\ \frac{[2r]_q (q^{2r} + (-1)^{r-1} + q^{-2r})}{r} & \text{if } (X_{\tilde{n}}^{(k)}, i, j) = (A_{2n}^{(2)}, 1, 1) \\ \frac{[\tilde{r}a_{ij}]_{q_i}}{\tilde{r}} & \text{otherwise, with } \tilde{r} = \frac{r}{\tilde{d}_{i,j}}. \end{cases}$$

Proposition 15.

If \sim is as in remark 10, then $\forall i, j \in I_0$, $r \in \mathbb{Z}$ $\tilde{b}_{i\tilde{j}r} = b_{ijr}$. In particular $(\mathcal{H}\mathcal{X})_{\tilde{i}, r; \tilde{j}, s}$ and $(\mathcal{H}\mathcal{H})_{\tilde{i}, r; \tilde{j}, s}$ are equivalent to

$$[\mathcal{H}_{i,r}, \mathcal{X}_{j,s}^\pm] = \pm b_{ijr} C^{\frac{r \mp |r|}{2}} \mathcal{X}_{j, r+s}^\pm \quad \text{and} \quad [\mathcal{H}_{i,r}, \mathcal{H}_{j,s}] = \delta_{r+s, 0} b_{ijr} \frac{C^r - C^{-r}}{q_j - q_j^{-1}}.$$

Proof: If $k = 1$ the claim is trivial. Suppose now $k > 1$ (so that $X_{\tilde{n}}$ is simply laced) and notice that (see remark 9,i))

$$\tilde{b}_{i\tilde{j}r} = \frac{1}{r} \sum_{u=0}^{k-1} \left[\frac{rk \tilde{a}_{i\chi^u(j)}}{\tilde{d}d_i} \right]_{q_i} \omega^{ru};$$

moreover if $\tilde{d}_{ij} = k$ then either $\chi(\tilde{i}) = \tilde{i}$ or $\chi(\tilde{j}) = \tilde{j}$ so that $\tilde{a}_{\tilde{i}\chi^u(\tilde{j})} = \tilde{a}_{\tilde{i}\tilde{j}}$ for all u , $\tilde{r} = \frac{r}{k}$, $\tilde{d}_h = d_h \forall h \in I_0$, $\tilde{d} = k$ and, thanks to remark 10,ii),

$$\tilde{b}_{\tilde{i}\tilde{j}r} = \frac{1}{r} \left[\frac{rk\tilde{a}_{\tilde{i}\tilde{j}}}{\tilde{d}d_i} \right]_{q_i} \sum_{u=0}^{k-1} \omega^{ru} = \begin{cases} 0 & \text{if } k \nmid r \\ \frac{1}{\tilde{r}} \left[\frac{ra_{ij}}{\tilde{d}} \right]_{q_i} = \frac{[\tilde{r}a_{ij}]_{q_i}}{\tilde{r}} & \text{if } k|r. \end{cases}$$

If $(X_{\tilde{n}}^{(k)}, i, j) = (A_{2n}^{(2)}, 1, 1)$

$$\tilde{b}_{\tilde{i}\tilde{j}r} = \frac{1}{r} ([4r]_q + (-1)^r [-2r]_q) = \frac{1}{r} [2r]_q (q^{2r} + (-1)^{r-1} + q^{-2r}).$$

In the remaining cases $\tilde{a}_{\tilde{i}\chi^u(\tilde{j})} = 0$ when $k \nmid u$, $\tilde{r} = r$ and $\max\{d_i, d_j\} = \frac{k}{d}$, hence

$$\tilde{b}_{\tilde{i}\tilde{j}r} = \frac{1}{r} \left[\frac{kr\tilde{a}_{\tilde{i}\tilde{j}}}{\tilde{d}d_i} \right]_{q_i} = \frac{1}{r} [ra_{ij}]_{q_i} = \frac{1}{\tilde{r}} [\tilde{r}a_{ij}]_{q_i}.$$

In the next remarks as well as in all the paper the q -commutators play a fundamental role in simplifying the description of the elements and in the computations. We recall here their definition and simple properties (see also [Jn]).

Notation 16.

Given $v \in \mathbb{C}(q) \setminus \{0\}$ and a, b elements of a $\mathbb{C}(q)$ -algebra, the element $[a, b]_v$ is defined by $[a, b]_v = ab - vba$.

Remark 17.

Let $a, b, c \in \tilde{\mathcal{U}}_q^{Dr}$ and $u, v, w \in \mathbb{C}(q) \setminus \{0\}$. Then:

- i) $[a, b]_u = -u[b, a]_{u^{-1}}$;
- ii) $[[a, b]_u, b]_v = [[a, b]_v, b]_u = ab^2 - (u + v)bab + uvb^2a$;
- iii) $[[a, b]_u, c]_v = [a, [b, c]_{v/w}]_{uw} - u[b, [a, c]_w]_{v/uw}$.

If moreover $a \in \mathcal{U}_{q,\alpha}^{Dr}$, $b \in \mathcal{U}_{q,\beta}^{Dr}$ and $i \in I_0$ then:

- iv) $[k_i a, b]_u = k_i [a, b]_{q^{-(\alpha_i|\beta)}u}$;
- v) $[a, k_i b]_u = q^{-(\alpha|\alpha_i)} k_i [a, b]_{q^{(\alpha_i|\alpha)}u}$.

Remark 18.

Let $i', j' \in \tilde{I}$; then:

- i) $F_{i'j'}^\pm(u_1, u_2)$ and $G_{i'j'}^\pm(u_1, u_2)$ are homogeneous polynomials of the same degree d ;
- ii) $F_{\chi(i')j'}^\pm(u_1, u_2) = \omega^d F_{i'j'}^\pm(\omega^{-1}u_1, u_2)$, $F_{i'\chi(j')}^\pm(u_1, u_2) = F_{i'j'}^\pm(u_1, \omega^{-1}u_2)$;
- iii) $G_{\chi(i')j'}^\pm(u_1, u_2) = \omega^d G_{i'j'}^\pm(\omega^{-1}u_1, u_2)$, $G_{i'\chi(j')}^\pm(u_1, u_2) = G_{i'j'}^\pm(u_1, \omega^{-1}u_2)$;
- iv) the relations (\mathcal{XFG}^\pm) are compatible with the relations $(\mathcal{Z}\mathcal{X}^\pm)$:

$$(\mathcal{XFG}^\pm)_{\chi(i'), j'}(u_1, u_2) = \omega^d (\mathcal{XFG}^\pm)_{i', j'}(\omega^{-1}u_1, u_2),$$

$$(\mathcal{XFG}^\pm)_{i',\chi(j')}(u_1, u_2) = (\mathcal{XFG}^\pm)_{i',j'}(u_1, \omega^{-1}u_2).$$

Remark 19.

Let $i', j' \in \tilde{I}$ be such that $\tilde{a}_{i',\chi^r(j')} = 0$ for all $r \in \mathbb{Z}$; this is equivalent to the condition $a_{\bar{i}',\bar{j}'} = 0$. Then:

- i) $F_{i',j'}^\pm(u_1, u_2) = G_{i',j'}^\pm(u_1, u_2) = 1$;
- ii) the relation $(\mathcal{XFG}^\pm)_{i',j'}$ is equivalent to $[\mathcal{X}_{i'}^\pm(u_1), \mathcal{X}_{j'}^\pm(u_2)] = 0$, that is to

$$[\mathcal{X}_{i',r}^\pm, \mathcal{X}_{j',s}^\pm] = 0 \quad \forall r, s \in \mathbb{Z}.$$

Remark 20.

Let $i', j' \in \tilde{I}$.

i) The condition $(\tilde{\alpha}_{i'}|\tilde{\alpha}_{\chi^r(j')}) = (\tilde{\alpha}_{i'}|\tilde{\alpha}_{j'}) \neq 0 \quad \forall r \in \mathbb{Z}$ is equivalent to the conditions $a_{\bar{i}',\bar{j}'} \neq 0$, $\tilde{d}_{\bar{i}',\bar{j}'} = k$ and implies that $\tilde{d}_{\bar{i}',\bar{j}'}(\tilde{\alpha}_{i'}|\tilde{\alpha}_{j'}) = d_{\bar{i}'}a_{\bar{i}',\bar{j}'}$;

ii) the condition $\exists! r \in \mathbb{Z}/k\mathbb{Z}$ such that $(\tilde{\alpha}_{i'}|\tilde{\alpha}_{\chi^r(j')}) \neq 0$ is equivalent to the conditions $a_{\bar{i}',\bar{j}'} \neq 0$, $\tilde{d}_{\bar{i}',\bar{j}'} = 1$, $(X_{\tilde{n}}^{(k)}, \tilde{i}', \tilde{j}') \neq (A_{2n}^{(2)}, 1, 1)$; this condition implies that $(\tilde{\alpha}_{i'}|\tilde{\alpha}_{\chi^r(j')}) = d_{\bar{i}'}a_{\bar{i}',\bar{j}'}$.

Let $i, j \in I_0$ and choose \sim as in remark 10.

iii) If $a_{ij} \neq 0$ and $(X_{\tilde{n}}^{(k)}, i, j) \neq (A_{2n}^{(2)}, 1, 1)$ (that is, \tilde{i}, \tilde{j} satisfy the conditions of i) or ii) with $r = 0$) then

$$F_{\tilde{i},\tilde{j}}^\pm(u_1, u_2) = u_1^{\tilde{d}_{ij}} - q_i^{\pm a_{ij}} u_2^{\tilde{d}_{ij}}, \quad G_{\tilde{i},\tilde{j}}^\pm(u_1, u_2) = q_i^{\pm a_{ij}} u_1^{\tilde{d}_{ij}} - u_2^{\tilde{d}_{ij}}$$

and the relation $(\mathcal{XFG}^\pm)_{\tilde{i},\tilde{j}}$ is equivalent to

$$[u_1^{\tilde{d}_{ij}} \mathcal{X}_{\tilde{i}}^\pm(u_1), \mathcal{X}_{\tilde{j}}^\pm(u_2)]_{q_i^{\pm a_{ij}}} + [u_2^{\tilde{d}_{ij}} \mathcal{X}_{\tilde{j}}^\pm(u_2), \mathcal{X}_{\tilde{i}}^\pm(u_1)]_{q_i^{\pm a_{ij}}} = 0,$$

that is to

$$[\mathcal{X}_{\tilde{i},r+\tilde{d}_{ij}}^\pm, \mathcal{X}_{\tilde{j},s}^\pm]_{q_i^{\pm a_{ij}}} + [\mathcal{X}_{\tilde{j},s+\tilde{d}_{ij}}^\pm, \mathcal{X}_{\tilde{i},r}^\pm]_{q_i^{\pm a_{ij}}} = 0 \quad \forall r, s \in \mathbb{Z}.$$

Notice that

$$[\mathcal{X}_{\tilde{i},r+\tilde{d}_{ij}}^-, \mathcal{X}_{\tilde{j},s}^-]_{q_i^{-a_{ij}}} = -q_i^{-a_{ij}} [\mathcal{X}_{\tilde{j},(s+\tilde{d}_{ij})-\tilde{d}_{ij}}^-, \mathcal{X}_{\tilde{i},r+\tilde{d}_{ij}}^-]_{q_i^{a_{ij}}}$$

so that $(\mathcal{XFG}^\pm)_{\tilde{i},\tilde{j}}$ is equivalent to

$$[\mathcal{X}_{\tilde{i},r\pm\tilde{d}_{ij}}^\pm, \mathcal{X}_{\tilde{j},s}^\pm]_{q_i^{a_{ij}}} + [\mathcal{X}_{\tilde{j},s\pm\tilde{d}_{ij}}^\pm, \mathcal{X}_{\tilde{i},r}^\pm]_{q_i^{a_{ij}}} = 0 \quad \forall r, s \in \mathbb{Z}.$$

Remark 21.

Let $(X_{\tilde{n}}^{(k)}, i, j) = (A_{2n}^{(2)}, 1, 1)$; then if \sim is as in remark 10:

- i) $F_{\tilde{i},\tilde{j}}^\pm(u_1, u_2) = (u_1 - q^{\pm 4}u_2)(u_1 + q^{\mp 2}u_2) = u_1^2 - (q^{\pm 4} - q^{\mp 2})u_1u_2 - q^{\pm 2}u_2^2$, $G_{\tilde{i},\tilde{j}}^\pm(u_1, u_2) = q^{\pm 2}u_1^2 - (q^{\pm 4} - q^{\mp 2})u_1u_2 - u_2^2$;

ii) the relation $(\mathcal{XFG}^\pm)_{\tilde{i}\tilde{j}}$ is equivalent to

$$\begin{aligned} & [u_1^2 \mathcal{X}_1^\pm(u_1), \mathcal{X}_1^\pm(u_2)]_{q^{\pm 2}} + [u_2^2 \mathcal{X}_1^\pm(u_2), \mathcal{X}_1^\pm(u_1)]_{q^{\pm 2}} + \\ & -(q^{\pm 4} - q^{\mp 2})(u_1 \mathcal{X}_1^\pm(u_1) u_2 \mathcal{X}_1^\pm(u_2) + u_2 \mathcal{X}_1^\pm(u_2) u_1 \mathcal{X}_1^\pm(u_1)) = 0, \end{aligned}$$

that is to

$$\begin{aligned} & [\mathcal{X}_{1,r+2}^\pm, \mathcal{X}_{1,s}^\pm]_{q^{\pm 2}} - q^{\pm 4} [\mathcal{X}_{1,r+1}^\pm, \mathcal{X}_{1,s+1}^\pm]_{q^{\mp 6}} + \\ & + [\mathcal{X}_{1,s+2}^\pm, \mathcal{X}_{1,r}^\pm]_{q^{\pm 2}} - q^{\pm 4} [\mathcal{X}_{1,s+1}^\pm, \mathcal{X}_{1,r+1}^\pm]_{q^{\mp 6}} = 0 \quad \forall r, s \in \mathbb{Z}. \end{aligned}$$

As in remark 20 notice that in this case $(\mathcal{XFG}^\pm)_{\tilde{i}\tilde{j}}$ is equivalent to

$$\begin{aligned} & [\mathcal{X}_{1,r\pm 2}^\pm, \mathcal{X}_{1,s}^\pm]_{q^2} - q^4 [\mathcal{X}_{1,r\pm 1}^\pm, \mathcal{X}_{1,s\pm 1}^\pm]_{q^{-6}} + \\ & + [\mathcal{X}_{1,s\pm 2}^\pm, \mathcal{X}_{1,r}^\pm]_{q^2} - q^4 [\mathcal{X}_{1,s\pm 1}^\pm, \mathcal{X}_{1,r\pm 1}^\pm]_{q^{-6}} = 0 \quad \forall r, s \in \mathbb{Z}. \end{aligned}$$

Remark 22.

Let $i' \in \tilde{I}$;

i) the relations $(\mathcal{X}3^{\varepsilon,\pm})$ are compatible with relations $(\mathcal{Z}\mathcal{X}^\pm)$:

$$(\mathcal{X}3^{\varepsilon,\pm})_{\chi(i')}(u_1, u_2, u_3) = (\mathcal{X}3^{\varepsilon,\pm})_{i'}(\omega^{-1}u_1, \omega^{-1}u_2, \omega^{-1}u_3);$$

ii) the condition $\tilde{a}_{\chi(i')i'} = -1$ is equivalent to the condition $(X_n^{(k)}, \vec{i}') = (A_{2n}^{(k)}, 1)$;

iii) the relations $(\mathcal{X}3^{\varepsilon,\pm})$ are equivalent to

$$\sum_{\sigma \in \mathcal{S}_3} \sigma.((q^{-3\varepsilon} u_1^{\pm\varepsilon} \mathcal{X}_{i'}^\pm(u_1) \mathcal{X}_{i'}^\pm(u_2) \mathcal{X}_{i'}^\pm(u_3) - q^{-\varepsilon} \mathcal{X}_{i'}^\pm(u_2) u_1^{\pm\varepsilon} \mathcal{X}_{i'}^\pm(u_1) \mathcal{X}_{i'}^\pm(u_3) +$$

$$- q^\varepsilon \mathcal{X}_{i'}^\pm(u_3) u_1^{\pm\varepsilon} \mathcal{X}_{i'}^\pm(u_1) \mathcal{X}_{i'}^\pm(u_2) + q^{3\varepsilon} \mathcal{X}_{i'}^\pm(u_3) \mathcal{X}_{i'}^\pm(u_2) u_1^{\pm\varepsilon} \mathcal{X}_{i'}^\pm(u_1)) = 0,$$

which is

$$q^{-3\varepsilon} \sum_{\sigma \in \mathcal{S}_3} \sigma. [[u_1^{\pm\varepsilon} \mathcal{X}_{i'}^\pm(u_1), \mathcal{X}_{i'}^\pm(u_2)]_{q^{2\varepsilon}}, \mathcal{X}_{i'}^\pm(u_3)]_{q^{4\varepsilon}} = 0$$

or equivalently

$$\sum_{\sigma \in \mathcal{S}_3} \sigma. [[\mathcal{X}_{i',r_1\pm\varepsilon}^\pm, \mathcal{X}_{i',r_2}^\pm]_{q^{2\varepsilon}}, \mathcal{X}_{i',r_3}^\pm]_{q^{4\varepsilon}} = 0 \quad \forall r_1, r_2, r_3 \in \mathbb{Z}.$$

Remark 23.

The relations (\mathcal{S}^\pm) are compatible with $(\mathcal{Z}\mathcal{X}^\pm)$.

Remark 24.

Let $k > 1$, $i', j' \in \tilde{I}$ be such that $\chi(i') \neq j'$, $\tilde{a}_{i'j'} < 0$ (this is equivalent to the condition $k > 1$, $a_{\tilde{i}'\tilde{j}'} < 0$). It is immediate to see that:

$$\text{i) } P_{\chi(i')j'}^\pm(u_1, u_2) = P_{i'j'}^\pm(u_1, u_2) = P_{i'\chi(j')}^\pm(u_1, u_2);$$

$$\text{ii) } m_{\chi(i')j'} = m_{i'j'} = m_{i'\chi(j')};$$

- iii) $P_{i',j'}^\pm(u_1, u_2)$ is homogeneous (of some degree d);
- iv) the relations (\mathcal{XP}^\pm) are compatible with (\mathcal{ZX}^\pm) :

$$(\mathcal{XP}^\pm)_{\chi(i'),j'}(u_1, u_2; v) = \omega^d (\mathcal{XP}^\pm)_{i',j'}(\omega^{-1}u_1, \omega^{-1}u_2; v)$$

and

$$(\mathcal{XP}^\pm)_{\chi(i'),j'}(u_1, u_2; \omega^{-1}v) = (\mathcal{XP}^\pm)_{i',j'}(u_1, u_2; v).$$

Moreover if $i, j \in I_0$ are such that $i = \bar{i}', j = \bar{j}'$ then:

- v) the condition $\tilde{a}_{i',\chi(i')} = 0$ and $\chi(j') \neq j'$, or $\chi(i') = i'$ is equivalent to the condition $a_{ij} = -1$;
- vi) $m_{i',j'} = \begin{cases} d_i & \text{if } a_{ij} = -1 \\ k(= -a_{ij}) & \text{otherwise;} \end{cases}$
- vii) the relation $(\mathcal{XP}^\pm)_{i',j'}$ is equivalent to

$$\begin{aligned} \sum_{\sigma \in \mathcal{S}_2} \sigma. \sum_{\substack{r,s \geq 0 \\ r+s = -1-a_{ij}}} q^{\pm 2s} (\mathcal{X}_{j',v}^\pm(u_1) u_1^s \mathcal{X}_{i'}^\pm(u_1) u_2^r \mathcal{X}_{i'}^\pm(u_2) + \\ - [2]_q m_{i',j'} u_1^s \mathcal{X}_{i'}^\pm(u_1) \mathcal{X}_{j',v}^\pm(u_1) u_2^r \mathcal{X}_{i'}^\pm(u_2) + \\ + u_1^s \mathcal{X}_{i'}^\pm(u_1) u_2^r \mathcal{X}_{i'}^\pm(u_2) \mathcal{X}_{j',v}^\pm(u_1)) = 0 \end{aligned}$$

that is

$$\begin{aligned} \sum_{\sigma \in \mathcal{S}_2} \sigma. \sum_{\substack{r,s \geq 0 \\ r+s = -1-a_{ij}}} q^{\pm 2s} (\mathcal{X}_{j',v}^\pm \mathcal{X}_{i',p_1+s}^\pm \mathcal{X}_{i',p_2+r}^\pm + \\ - [2]_q m_{i',j'} \mathcal{X}_{i',p_1+s}^\pm \mathcal{X}_{j',v}^\pm \mathcal{X}_{i',p_2+r}^\pm + \mathcal{X}_{i',p_1+s}^\pm \mathcal{X}_{i',p_2+r}^\pm \mathcal{X}_{j',v}^\pm) = 0, \end{aligned}$$

or equivalently

$$\begin{aligned} \sum_{\sigma \in \mathcal{S}_2} \sigma. \sum_{\substack{r,s \geq 0 \\ r+s = -1-a_{ij}}} q^{2s} (\mathcal{X}_{j',v}^\pm \mathcal{X}_{i',p_1 \pm s}^\pm \mathcal{X}_{i',p_2 \pm r}^\pm + \\ - [2]_q m_{i',j'} \mathcal{X}_{i',p_1 \pm s}^\pm \mathcal{X}_{j',v}^\pm \mathcal{X}_{i',p_2 \pm r}^\pm + \mathcal{X}_{i',p_1 \pm s}^\pm \mathcal{X}_{i',p_2 \pm r}^\pm \mathcal{X}_{j',v}^\pm) = 0. \end{aligned}$$

We are now ready to write down an equivalent definition of $\mathcal{U}_q^{Dr}(X_n^{(k)})$, using the generators (G) .

Proposition 25.

$\mathcal{U}_q^{Dr}(X_n^{(k)})$ is (isomorphic to) the $\mathbb{C}(q)$ -algebra generated by (G)
 $C^{\pm 1}, k_i^{\pm 1} (i \in I_0), X_{i,r}^\pm ((i, r) \in I_0 \times \mathbb{Z}), H_{i,r} ((i, r) \in I_0 \times (\mathbb{Z} \setminus \{0\}))$,
 with the following relations (DR) :

$$(ZX^\pm) \quad X_{i,r}^\pm = 0 \quad \forall (i, r) \in (I_0 \times \mathbb{Z}) \setminus I_\mathbb{Z},$$

$$(ZH) \quad H_{i,r} = 0 \quad \forall (i, r) \in (I_0 \times \mathbb{Z}) \setminus I_\mathbb{Z},$$

$$\begin{aligned}
 (CUK) \quad & [C, x] = 0 \quad \forall x, \quad k_i k_j = k_j k_i \quad (i, j \in I_0), \\
 (CK) \quad & CC^{-1} = 1, \quad k_i k_i^{-1} = 1 = k_i^{-1} k_i \quad (i \in I_0), \\
 (KX^\pm) \quad & k_i X_{j,r}^\pm = q_i^{\pm a_{ij}} X_{j,r}^\pm k_i \quad (i \in I_0, (j, r) \in I_0 \times \mathbb{Z}), \\
 (KH) \quad & [k_i, H_{j,s}] = 0 \quad (i \in I_0, (j, s) \in I_0 \times (\mathbb{Z} \setminus \{0\})), \\
 (XX) \quad & [X_{i,r}^+, X_{j,s}^-] = \begin{cases} \delta_{i,j} \frac{C^{-s} k_i \tilde{H}_{i,r+s}^+ - C^{-r} k_i^{-1} \tilde{H}_{i,r+s}^-}{q_i - q_i^{-1}} & \text{if } \tilde{d}_j | s \\ 0 & \text{otherwise} \end{cases} \\
 & ((i, r), (j, s) \in I_0 \times \mathbb{Z}) \\
 (HX^\pm) \quad & [H_{i,r}, X_{j,s}^\pm] = \pm b_{ijr} C^{\frac{r \mp |r|}{2}} X_{j,r+s}^\pm \quad ((i, r) \in I_0 \times (\mathbb{Z} \setminus \{0\}), (j, s) \in I_0 \times \mathbb{Z}), \\
 (HH) \quad & [H_{i,r}, H_{j,s}] = \delta_{r+s,0} b_{ijr} \frac{C^r - C^{-r}}{q_j - q_j^{-1}} \quad ((i, r), (j, s) \in I_0 \times (\mathbb{Z} \setminus \{0\})), \\
 (XD^\pm) \quad & [X_{i,r \pm \tilde{d}_{ij}}^\pm, X_{j,s}^\pm]_{q_i^{a_{ij}}} + [X_{j,s \pm \tilde{d}_{ij}}^\pm, X_{i,r}^\pm]_{q_j^{a_{ji}}} = 0 \quad ((i, r), (j, s) \in I_0 \times \mathbb{Z}, a_{ij} < 0), \\
 (X1^\pm) \quad & \sum_{\sigma \in \mathcal{S}_2} \sigma. [X_{i,r_1 \pm \tilde{d}_i}^\pm, X_{i,r_2}^\pm]_{q_i^2} = 0 \quad ((r_1, r_2) \in \mathbb{Z}^2, (X_{\tilde{n}}^{(k)}, i) \neq (A_{2n}^{(2)}, 1)), \\
 (X2^\pm) \quad & \sum_{\sigma \in \mathcal{S}_2} \sigma. ([X_{1,r_1 \pm 2}^\pm, X_{1,r_2}^\pm]_{q^2} - q^4 [X_{1,r_1 \pm 1}^\pm, X_{1,r_2 \pm 1}^\pm]_{q^{-6}}) = 0 \\
 & ((r_1, r_2) \in \mathbb{Z}^2, X_{\tilde{n}}^{(k)} = A_{2n}^{(2)}), \\
 (X3^{\varepsilon, \pm}) \quad & \sum_{\sigma \in \mathcal{S}_3} \sigma. [[X_{1,r_1 \pm \varepsilon}^\pm, X_{1,r_2}^\pm]_{q^{2\varepsilon}}, X_{1,r_3}^\pm]_{q^{4\varepsilon}} = 0 \\
 & ((r_1, r_2, r_3) \in \mathbb{Z}^3, X_{\tilde{n}}^{(k)} = A_{2n}^{(2)}), \\
 (SUL^\pm) \quad & \sum_{\sigma \in \mathcal{S}_{1-a_{ij}}} \sigma. \sum_{u=0}^{1-a_{ij}} (-1)^u \begin{bmatrix} 1-a_{ij} \\ u \end{bmatrix}_{q_i} X_{i,r_1}^\pm \cdots X_{i,r_u}^\pm X_{j,s}^\pm X_{i,r_{u+1}}^\pm \cdots X_{i,r_{1-a_{ij}}}^\pm = 0 \\
 & (i \neq j \in I_0, a_{ij} \in \{0, -1\} \text{ if } k \neq 1, r = (r_1, \dots, r_{1-a_{ij}}) \in \mathbb{Z}^{1-a_{ij}}, s \in \mathbb{Z}), \\
 (S2^\pm) \quad & \sum_{\sigma \in \mathcal{S}_2} \sigma. (q(X_{j,s}^\pm X_{i,r_1 \pm 1}^\pm X_{i,r_2}^\pm - [2]_{q^2} X_{i,r_1 \pm 1}^\pm X_{j,s}^\pm X_{i,r_2}^\pm + X_{i,r_1 \pm 1}^\pm X_{i,r_2}^\pm X_{j,s}^\pm) + \\
 & + q^{-1} (X_{j,s}^\pm X_{i,r_1}^\pm X_{i,r_2 \pm 1}^\pm - [2]_{q^2} X_{i,r_1}^\pm X_{j,s}^\pm X_{i,r_2 \pm 1}^\pm + X_{i,r_1}^\pm X_{i,r_2 \pm 1}^\pm X_{j,s}^\pm)) = 0
 \end{aligned}$$

$$(i, j \in I_0, a_{ij} = -2, k = 2, (r_1, r_2) \in \mathbb{Z}^2, s \in \mathbb{Z}),$$

$$(S3^\pm) \quad \sum_{\sigma \in S_2} \sigma \cdot \left(q^2 (X_{j,s}^\pm X_{i,r_1 \pm 2}^\pm X_{i,r_2}^\pm - [2]_{q^3} X_{i,r_1 \pm 2}^\pm X_{j,s}^\pm X_{i,r_2}^\pm + \right. \\ \left. + X_{i,r_1 \pm 2}^\pm X_{i,r_2}^\pm X_{j,s}^\pm) + (X_{j,s}^\pm X_{i,r_1 \pm 1}^\pm X_{i,r_2 \pm 1}^\pm - [2]_{q^3} X_{i,r_1 \pm 1}^\pm X_{j,s}^\pm X_{i,r_2 \pm 1}^\pm + \right. \\ \left. + X_{i,r_1 \pm 1}^\pm X_{i,r_2 \pm 1}^\pm X_{j,s}^\pm) + q^{-2} (X_{j,s}^\pm X_{i,r_1}^\pm X_{i,r_2 \pm 2}^\pm + \right. \\ \left. - [2]_{q^3} X_{i,r_1}^\pm X_{j,s}^\pm X_{i,r_2 \pm 2}^\pm + X_{i,r_1}^\pm X_{i,r_2 \pm 2}^\pm X_{j,s}^\pm) \right) = 0 \\ (i, j \in I_0, a_{ij} = -3, k = 3, (r_1, r_2) \in \mathbb{Z}^2, s \in \mathbb{Z}),$$

where $\varepsilon \in \{\pm 1\}$ and $\tilde{H}_{i,r}^\pm$ and b_{ijr} are defined as follows:

$$\sum_{r \in \mathbb{Z}} \tilde{H}_{i,\pm r}^\pm u^r = \exp \left(\pm (q_i - q_i^{-1}) \sum_{r > 0} H_{i,\pm r} u^r \right); \\ b_{ijr} = \begin{cases} 0 & \text{if } \tilde{d}_{i,j} \nmid r \\ \frac{[2r]_q (q^{2r} + (-1)^{r-1} + q^{-2r})}{r} & \text{if } (X_{\tilde{n}}^{(k)}, i, j) = (A_{2n}^{(2)}, 1, 1) \\ \frac{[\tilde{r} a_{ij}]_{q_i}}{\tilde{r}} & \text{otherwise, with } \tilde{r} = \frac{r}{d_{i,j}}. \end{cases}$$

An isomorphism is given by

$$C^{\pm 1} \mapsto \mathcal{C}^{\pm 1}, \quad k_i^{\pm 1} \mapsto \mathcal{K}_{\tilde{i}}^{\pm 1}, \quad X_{i,r}^\pm \mapsto \mathcal{X}_{i,r}^\pm, \quad H_{i,s} \mapsto \mathcal{H}_{\tilde{i},s}$$

where $(i, r), (i, s) \in I_0 \times \mathbb{Z}$ ($s \neq 0$) and $\tilde{\cdot} : I_0 \rightarrow \tilde{I}$ is a section as in remark 10; its inverse is

$$\mathcal{C}^{\pm 1} \mapsto C^{\pm 1}, \quad \mathcal{K}_{i'}^{\pm 1} \mapsto k_{i'}^{\pm 1}, \quad \mathcal{X}_{\chi^u(\tilde{i}),r}^\pm \mapsto \omega^{ur} X_{i,r}^\pm, \quad \mathcal{H}_{\chi^u(\tilde{i}),s} \mapsto \omega^{us} H_{i,s}$$

($i' \in \tilde{I}$, $i \in I_0$, $u, r \in \mathbb{Z}$, $s \in \mathbb{Z} \setminus \{0\}$).

Proof: The claim follows from remarks 4, corollary 6, remarks 11-13, proposition 15 and remarks 18-24.

Remark 26.

$\mathcal{U}_q^{Dr}(X_{\tilde{n}}^{(k)})$ is (isomorphic to) the $\mathbb{C}(q)$ -algebra generated by (G')
 $C^{\pm 1}, \quad k_i^{\pm 1} \quad (i \in I_0), \quad X_{i,r}^\pm \quad ((i, r) \in I_{\mathbb{Z}}), \quad H_{i,r} \quad ((i, r) \in I_{\mathbb{Z}} \setminus (I_0 \times \{0\})),$

with relations (CUK') -($S3^\pm$), where, for a relation (R) , the relation (R') is the set of relations in (R) whose left hand side does not involve indices in $(I_0 \times \mathbb{Z}) \setminus I_{\mathbb{Z}}$.

Remark that the only case where the right hand side of some relation in (R') involves indices in $(I_0 \times \mathbb{Z}) \setminus I_{\mathbb{Z}}$ is the case $(R) = (HX^\pm)$: in

this situation if $(j, r + s) \notin I_{\mathbb{Z}}$ then $\tilde{d}_j \nmid r$ and $b_{ijr} = 0$, hence (HX'^{\pm}) is the following relation:

$$[H_{i,r}, X_{j,s}^{\pm}] = \begin{cases} 0 & \text{if } \tilde{d}_j \nmid r \\ \pm b_{ijr} C^{\frac{r \mp |r|}{2}} X_{j,r+s}^{\pm} & \text{if } \tilde{d}_j \mid r \end{cases} \quad ((i, r) \in I_{\mathbb{Z}} \setminus (I_0 \times \{0\}), (j, s) \in I_{\mathbb{Z}}).$$

Remark 27.

Since in the $\mathbb{C}(q)$ -algebra generated by (G) for any of the relations (R) defining $\mathcal{U}_q^{Dr}(X_{\tilde{n}}^{(k)})$ the relations (ZX, ZH, R) are equivalent to the relations (ZX, ZH, R') , by abuse of notation we shall denote by (R) also the relation (R') .

It is with the presentation of \mathcal{U}_q^{Dr} given in proposition 25 that we shall deal from now on.

§5. MORE about the DEFINITION of \mathcal{U}_q^{Dr} .

The material of this section is presented in order to simplify the exposition and to handle more easily the relations defining \mathcal{U}_q^{Dr} , with the aim of sharply reducing them: some notations will be fixed; a new formulation will be given, mainly in terms of q -commutators, of some of the relations of proposition §4.25; and some new relations $((T2^{\pm})$ and $(T3^{\pm}))$ will be introduced and proved to be equivalent, under suitable conditions, to $(S2^{\pm})$ and $(S3^{\pm})$. Also the Serre relations are introduced here, but they will be studied in details in section §11.

Notation 1.

Let \mathcal{U} be an algebra and let (R) denote the relations

$$(R) \quad S_{\zeta}(r, s) = 0 \quad (\zeta \in \mathcal{Z}, r \in \mathbb{Z}^l, s \in \mathbb{Z}^{\tilde{l}}),$$

where \mathcal{Z} is a set, $l \in \mathbb{Z}_+$, $\tilde{l} \in \{0, 1\}$, $S_{\zeta}(r, s) \in \mathcal{U}$. Then:

i) for all $\zeta \in \mathcal{Z}$, denote by (R_{ζ}) the relations

$$(R_{\zeta}) \quad S_{\zeta}(r, s) = 0 \quad (r \in \mathbb{Z}^l, s \in \mathbb{Z}^{\tilde{l}});$$

of course if $\#\mathcal{Z} = 1$ and $\mathcal{Z} = \{\zeta\}$ then $(R) = (R_{\zeta})$;

ii) denote by $\mathcal{I}(R)$ the ideal of \mathcal{U} generated by the $S_{\zeta}(r, s)$'s:

$$\mathcal{I}(R) = (S_{\zeta}(r, s) | \zeta \in \mathcal{Z}, r \in \mathbb{Z}^l, s \in \mathbb{Z}^{\tilde{l}});$$

of course $\mathcal{I}(R) = (\mathcal{I}(R_{\zeta}) | \zeta \in \mathcal{Z})$;

iii) if $(^{(h)}R)$ ($h = 1, \dots, m$) are the relations

$$(^{(h)}R) \quad (^{(h)}S_{\zeta}(r, s) = 0 \quad (\zeta \in ^{(h)}\mathcal{Z}, r \in \mathbb{Z}^{l_h}, s \in \mathbb{Z}^{\tilde{l}_h}),$$

where $^{(h)}\mathcal{Z}$ is a set, $l_h \in \mathbb{Z}_+$, $\tilde{l}_h \in \{0, 1\}$, $^{(h)}S_{\zeta}^{\pm}(r, s) \in \mathcal{U}$, define

$$\mathcal{I}(^{(1)}R, \dots, ^{(m)}R) = (\mathcal{I}(^{(1)}R), \dots, \mathcal{I}(^{(m)}R))$$

iv) if (R^{\pm}) denotes the relations

$$(R^{\pm}) \quad S_{\zeta}^{\pm}(r, s) = 0 \quad (\zeta \in \mathcal{Z}, r \in \mathbb{Z}^l, s \in \mathbb{Z}^{\tilde{l}}),$$

where \mathcal{Z} is a set, $l \in \mathbb{Z}_+$, $\tilde{l} \in \{0, 1\}$, $S_{\zeta}^{\pm}(r, s) \in \mathcal{U}$, denote by (R) the relations

$$(R) \quad S_{\zeta'}(r, s) = 0 \quad (\zeta' \in \mathcal{Z} \times \{\pm\}, r \in \mathbb{Z}^l, s \in \mathbb{Z}^{\tilde{l}}),$$

where $S_{(\zeta, \pm)}(r, s) = S_{\zeta}^{\pm}(r, s)$; in particular

$$\mathcal{I}(R) = (\mathcal{I}(R^+), \mathcal{I}(R^-));$$

moreover denote by $\mathcal{I}^{\pm}(R)$ the ideals

$$\mathcal{I}^+(R) = \mathcal{I}(R^+) \quad \text{and} \quad \mathcal{I}^-(R) = \mathcal{I}(R^-).$$

Notation 2.

For $i, j \in I_0$, $l \in \mathbb{N}$, $a \in \mathbb{Z}$, $r = (r_1, \dots, r_l) \in \mathbb{Z}^l$, $s \in \mathbb{Z}$ we set

$$X_{i,j;l;a}^{\pm}(r; s) = \sum_{u=0}^l (-1)^u \begin{bmatrix} l \\ u \end{bmatrix}_{q_i^a} X_{i,r_1}^{\pm} \cdot \dots \cdot X_{i,r_u}^{\pm} X_{j,s}^{\pm} X_{i,r_{u+1}}^{\pm} \cdot \dots \cdot X_{i,r_l}^{\pm}.$$

Remark 3.

The relations (SUL^{\pm}) , $(S2^{\pm})$ and $(S3^{\pm})$ can be written in a more compact form as:

$$\sum_{\sigma \in \mathcal{S}_{1-a_{ij}}} \sigma \cdot X_{i,j;1-a_{ij};1}^{\pm}(r; s) = 0,$$

which is (SUL^{\pm}) , and

$$\sum_{\sigma \in \mathcal{S}_2} \sigma \cdot \sum_{\substack{u,v \geq 0 \\ u+v=-1-a_{ij}}} q^{v-u} X_{i,j;2;-a_{ij}}^{\pm}(r_1 \pm v, r_2 \pm u; s) = 0,$$

which is $(S2^{\pm})$, $(S3^{\pm})$ and also (SUL^{\pm}) in the case $a_{ij} = -1$.

In order to express the relations (SUL^{\pm}) in terms of q -commutators, and for further use and simplifications, we introduce the following notation.

Notation 4.

For $i \neq j \in I_0$, $l \in \mathbb{N}$, $a \in \mathbb{Z}$, $r = (r_1, \dots, r_l) \in \mathbb{Z}^l$, $s \in \mathbb{Z}$ set

$$M_{i,j;l;a}^\pm(r_1, \dots, r_l; s) = \begin{cases} X_{j,s}^\pm & \text{if } l = 0 \\ [M_{i,j;l-1;a}^\pm(r_1, \dots, r_{l-1}; s), X_{i,r_l}^\pm]_{q_i}^{-a_{ij}-2a(l-1)} & \text{if } l > 0. \end{cases}$$

Remark 5.

The relations (SUL^\pm) can be formulated in terms of q -commutators as

$$\sum_{\sigma \in S_{1-a_{ij}}} \sigma.M_{i,j;1-a_{ij};1}^\pm(r; s) = 0$$

($i \neq j \in I_0$, $a_{ij} \in \{0, -1\}$ if $k \neq 1$, $r \in \mathbb{Z}^{1-a_{ij}}$, $s \in \mathbb{Z}$).

Remark 6.

Also the relations $(S2^\pm)$ and $(S3^\pm)$ can be formulated in terms of q -commutators:

i) $(S2^\pm)$ can be written as:

$$\sum_{\sigma \in S_2} \sigma.((q^2 + q^{-2})[[X_{j,s}^\pm, X_{i,r_1 \pm 1}^\pm]_{q^2}, X_{i,r_2}^\pm] + q^2[[X_{i,r_1 \pm 1}^\pm, X_{i,r_2}^\pm]_{q^2}, X_{j,s}^\pm]_{q^{-4}}) = 0;$$

ii) moreover $(S2^+)$ can be written also in one of the following equivalent ways:

$$\sum_{\sigma \in S_2} \sigma.((q^2 + q^{-2})[[X_{j,s}^+, X_{i,r_1}^+]_{q^{-2}}, X_{i,r_2+1}^+] + [X_{j,s}^+, [X_{i,r_2+1}^+, X_{i,r_1}^+]_{q^2}]_{q^{-4}}) = 0;$$

$$\sum_{\sigma \in S_2} \sigma.([X_{j,s}^+, X_{i,r_1+1}^+]_{q^{-2}}, X_{i,r_2}^+] - q^2[X_{i,r_1+1}^+, [X_{j,s}^+, X_{i,r_2}^+]_{q^{-2}}]_{q^{-4}}) = 0;$$

iii) $(S3^\pm)$ can be formulated in terms of q -commutators as follows:

$$\begin{aligned} & \sum_{\sigma \in S_2} \sigma.((q^2 + q^{-4})[[X_{j,s}^\pm, X_{i,r_1 \pm 2}^\pm]_{q^3}, X_{i,r_2}^\pm]_{q^{-1}} + \\ & + (1 - q^{-2} + q^{-4})[[X_{j,s}^\pm, X_{i,r_1 \pm 1}^\pm]_{q^3}, X_{i,r_2 \pm 1}^\pm]_{q^4} + \\ & + q^2[[X_{i,r_1 \pm 2}^\pm, X_{i,r_2}^\pm]_{q^2} + [X_{i,r_2 \pm 1}^\pm, X_{i,r_1 \pm 1}^\pm]_{q^2}, X_{j,s}^\pm]_{q^{-6}}) = 0; \end{aligned}$$

Definition 7.

Consider the case $k > 1$, $X_{\tilde{n}}^{(k)} \neq A_{2n}^{(2)}$ and introduce the relations

(Tk^\pm) :

$(T2^\pm)$

$$\sum_{\sigma \in S_2} \sigma.([X_{j,s}^\pm, X_{i,r_1 \pm 1}^\pm]_{q^2}, X_{i,r_2}^\pm] = 0 \quad (i, j \in I_0, a_{ij} = -2, r \in \mathbb{Z}^2, s \in \mathbb{Z});$$

$$(T3^\pm) \sum_{\sigma \in \mathcal{S}_2} \sigma.((q^2+1)[[X_{j,s}^\pm, X_{i,r_1 \pm 2}^\pm]_{q^3}, X_{i,r_2}^\pm]_{q^{-1}} + [[X_{j,s}^\pm, X_{i,r_1 \pm 1}^\pm]_{q^3}, X_{i,r_2 \pm 1}^\pm]_q) = 0$$

$$(i, j \in I_0, a_{ij} = -3, r \in \mathbb{Z}^2, s \in \mathbb{Z}).$$

Proposition 8.

Let $k > 1$, $X_n^{(k)} \neq A_{2n}^{(2)}$.

Then $\mathcal{I}(X1^\pm, Sk^\pm) = \mathcal{I}(X1^\pm, Tk^\pm)$.

More precisely if $i, j \in I_0$ are such that $a_{ij} < -1$ we have that

$$\mathcal{I}(X1_i^\pm, Sk^\pm) = \mathcal{I}(X1_i^\pm, Tk^\pm) \quad (\text{see notation } 1, i).$$

In particular, $(S2^\pm)$ and $(S3^\pm)$ can be replaced respectively by $(T2^\pm)$ and $(T3^\pm)$ among the defining relations of \mathcal{U}_q^{Dr} .

Proof: It is enough to notice that

$$\left[\sum_{\sigma \in \mathcal{S}_2} \sigma. [X_{i,r_1 \pm 1}^\pm, X_{i,r_2}^\pm]_{q^2}, X_{j,s}^\pm \right]_{q^{-4}}$$

and

$$[[X_{i,r_1 \pm 2}^\pm, X_{i,r_2}^\pm]_{q^2} + [X_{i,r_2 \pm 1}^\pm, X_{i,r_1 \pm 1}^\pm]_{q^2}, X_{j,s}^\pm]_{q^{-6}}$$

belong to $\mathcal{I}(X1_i^\pm)$.

Definition 9.

We recall also the Serre relations

$$(S^\pm) \sum_{\sigma \in \mathcal{S}_{1-a_{ij}}} \sigma. X_{i,j;1-a_{ij};1}^\pm(r; s) = 0 \quad (i \neq j \in I_0, r \in \mathbb{Z}^{1-a_{ij}}, s \in \mathbb{Z}).$$

Remark 10.

The Serre relations can be formulated in terms of q -commutators as

$$\sum_{\sigma \in \mathcal{S}_{1-a_{ij}}} \sigma. M_{i,j;1-a_{ij};1}^\pm(r; s) = 0 \quad (i \neq j \in I_0, r \in \mathbb{Z}^{1-a_{ij}}, s \in \mathbb{Z}).$$

Remark 11.

The right hand sides of relations (Tk^\pm) and (S^\pm) are zero, hence remark §4.27 holds for these relations (see also remark §4.26).

The comparison of the defining relations of \mathcal{U}_q^{Dr} with the Serre relations is the matter of section §11.

Notation 12.

Let us introduce also the following notations:

i) for $i, j \in I_0, r, s \in \mathbb{Z}$

$$M_{(2)}^\pm((i, r), (j, s)) = [X_{i,r \pm \tilde{d}_{ij}}^\pm, X_{j,s}^\pm]_{q_i^{a_{ij}}} + [X_{j,s \pm \tilde{d}_{ij}}^\pm, X_{i,r}^\pm]_{q_j^{a_{ji}}};$$

ii) for $i \in I_0$, $r = (r_1, r_2) \in \mathbb{Z}^2$

$$M_i^\pm(r) = [X_{i,r_1 \pm \tilde{d}_i}^\pm, X_{i,r_2}^\pm]_{q_i^2};$$

iii) if $X_{\tilde{n}}^{(k)} = A_{2n}^{(2)}$ and $r = (r_1, r_2) \in \mathbb{Z}^2$

$$M_{(2,2)}^\pm(r) = [X_{1,r_1 \pm 2}^\pm, X_{1,r_2}^\pm]_{q^2} - q^4 [X_{1,r_1 \pm 1}^\pm, X_{1,r_2 \pm 1}^\pm]_{q^{-6}};$$

iv) if $X_{\tilde{n}}^{(k)} = A_{2n}^{(2)}$ and $r = (r_1, r_2, r_3) \in \mathbb{Z}^3$

$$M_{(3)}^{\varepsilon, \pm}(r) = [[X_{1,r_1 \pm \varepsilon}^\pm, X_{1,r_2}^\pm]_{q^{2\varepsilon}}, X_{1,r_3}^\pm]_{q^{4\varepsilon}};$$

v) if $k > 1$ and $r = (r_1, r_2) \in \mathbb{Z}^2$, $s \in \mathbb{Z}$

$$X_{[k]}^\pm(r; s) = \sum_{\substack{u, v \geq 0 \\ u+v=k-1}} q^{v-u} X_{i,j;2;k}^\pm(r_1 \pm v, r_2 \pm u; s)$$

where $i, j \in I_0$ are such that $a_{ij} = -k$;

vi) if $k = 2$, $X_{\tilde{n}}^{(k)} \neq A_{2n}^{(2)}$ and $r = (r_1, r_2) \in \mathbb{Z}^2$, $s \in \mathbb{Z}$

$$M_{[2]}^\pm(r; s) = M_{i,j;2;1}^\pm(r_1 \pm 1, r_2; s)$$

where $i, j \in I_0$ are such that $a_{ij} = -2$;

vii) if $k = 3$ and $r = (r_1, r_2) \in \mathbb{Z}^2$, $s \in \mathbb{Z}$

$$M_{[3]}^\pm(r; s) = (q^2 + 1)M_{i,j;2;2}^\pm(r_1 \pm 2, r_2; s) + M_{i,j;2;1}^\pm(r_1 \pm 1, r_2 \pm 1; s)$$

where $i, j \in I_0$ are such that $a_{ij} = -3$.

Remark 13.

Of course the following relations depend on (ZX^\pm) :

i) $M_{(2)}^\pm((i, r), (j, s)) = 0$ if $(r, s) \notin \tilde{d}_i \mathbb{Z} \times \tilde{d}_j \mathbb{Z}$;

ii) $M_i^\pm(r) = 0$ if $r \notin (\tilde{d}_i \mathbb{Z})^2$;

iii) $M_{i,j;l;a}^\pm(r; s) = 0$ if $(r, s) \notin (\tilde{d}_i \mathbb{Z})^l \times \tilde{d}_j \mathbb{Z}$;

iv) $X_{i,j;l;a}^\pm(r; s) = 0$ if $(r, s) \notin (\tilde{d}_i \mathbb{Z})^l \times \tilde{d}_j \mathbb{Z}$;

v) $X_{[k]}^\pm(r; s) = 0$ if $s \notin \tilde{d} \mathbb{Z}$;

vi) $M_{[k]}^\pm(r; s) = 0$ if $s \notin k \mathbb{Z}$.

Remark 14.

Recalling remark §4.27 (and remark 13) we have the following obvious reformulation of the relations (XD^\pm) -($S3^\pm$), $(T2^\pm)$, $(T3^\pm)$ and (S^\pm) in terms of the notations just introduced (notations 2, 4, 12):

$$(XD^\pm) : \quad M_{(2)}^\pm((i, \tilde{d}_i r), (j, \tilde{d}_j s)) = 0 \quad (i, j \in I_0, a_{ij} < 0, r, s \in \mathbb{Z});$$

$$(X1^\pm) : \quad \sum_{\sigma \in \mathcal{S}_2} \sigma.M_i^\pm(\tilde{d}_i r) = 0 \quad (i \in I_0, (X_{\tilde{n}}^{(k)}, i) \neq (A_{2n}^{(2)}, i), r \in \mathbb{Z}^2);$$

$$(X2^\pm) : \quad \sum_{\sigma \in \mathcal{S}_2} \sigma.M_{(2,2)}^\pm(r) = 0 \quad (r \in \mathbb{Z}^2);$$

$$(X3^{\varepsilon,\pm}) : \quad \sum_{\sigma \in \mathcal{S}_3} \sigma.M_{(3)}^{\varepsilon,\pm}(r) = 0 \quad (r \in \mathbb{Z}^3);$$

$$(S(UL)^\pm) : \quad \sum_{\sigma \in \mathcal{S}_{1-a_{ij}}} \sigma.M_{i,j;1-a_{ij};1}^\pm(\tilde{d}_i r; \tilde{d}_j s) = 0$$

or equivalently

$$\sum_{\sigma \in \mathcal{S}_{1-a_{ij}}} \sigma.X_{i,j;1-a_{ij};1}^\pm(\tilde{d}_i r; \tilde{d}_j s) = 0$$

$$(i \neq j \in I_0 \ (a_{ij} \in \{0, -1\} \text{ if } k \neq 1), \ r \in \mathbb{Z}^{1-a_{ij}}, \ s \in \mathbb{Z});$$

$$(Sk^\pm) : \quad \sum_{\sigma \in \mathcal{S}_2} \sigma.X_{[k]}^\pm(r; \tilde{d}s) = 0 \quad (r \in \mathbb{Z}^2, \ s \in \mathbb{Z});$$

$$(Tk^\pm) : \quad \sum_{\sigma \in \mathcal{S}_2} \sigma.M_{[k]}^{\pm 1}(r; ks) = 0 \quad (r \in \mathbb{Z}^2, \ s \in \mathbb{Z}).$$

§6. $\bar{\mathcal{U}}_q^{Dr}$ and its STRUCTURES.

In order to study the relations defining \mathcal{U}_q^{Dr} it is convenient to proceed by steps: the algebras $\bar{\mathcal{U}}_q^{Dr} = \bar{\mathcal{U}}_q^{Dr}(X_{\tilde{n}}^{(k)})$ and $\tilde{\mathcal{U}}_q^{Dr} = \tilde{\mathcal{U}}_q^{Dr}(X_{\tilde{n}}^{(k)})$ here defined are such that $\tilde{\mathcal{U}}_q^{Dr}$ is a quotient of $\bar{\mathcal{U}}_q^{Dr}$ and \mathcal{U}_q^{Dr} is a quotient of $\tilde{\mathcal{U}}_q^{Dr}$.

This section is devoted to introduce some important structures on $\bar{\mathcal{U}}_q^{Dr}$ (Q -gradation, homomorphisms between some of these algebras, automorphisms and antihomomorphisms of each of them), which will be proved to induce analogous structures on $\tilde{\mathcal{U}}_q^{Dr}$ (see also section §7) and, which is finally important, on \mathcal{U}_q^{Dr} (see also sections §7 and §8).

Some remarks point out the first (trivially) unnecessary relations: (ZH) and (KH) are redundant.

Definition 1.

We denote by:

- i) $\tilde{\mathcal{U}}_q^{Dr}(X_{\tilde{n}}^{(k)})$ the $\mathbb{C}(q)$ -algebra generated by (G) with relations

$$(ZX^\pm), (CUK), (CK), (KX^\pm), (XX) \text{ and}$$

$$(HXL^\pm)$$

$$[H_{i,r}, X_{j,s}^\pm] = \pm b_{ijr} C^{\frac{r \mp |r|}{2}} X_{j,r+s}^\pm \quad ((i,r), (j,s) \in I_{\mathbb{Z}}, \tilde{d}_i \leq |r| \leq \tilde{d}_{ij});$$

ii) $\bar{\mathcal{U}}_q^{Dr}(X_{\tilde{n}}^{(k)})$ the $\mathbb{C}(q)$ -algebra generated by

$$(\bar{G}) \quad C^{\pm 1}, \quad k_i^{\pm 1} \quad (i \in I_0), \quad X_{i,r}^{\pm} \quad ((i, r) \in I_0 \times \mathbb{Z})$$

with relations

$$(ZX^{\pm}), (CUK), (CK).$$

Remark 2.

$\mathcal{U}_q^{Dr}(X_{\tilde{n}}^{(k)})$ is obviously a quotient of $\tilde{\mathcal{U}}_q^{Dr}(X_{\tilde{n}}^{(k)})$.

We shall prove that also $\tilde{\mathcal{U}}_q^{Dr}(X_{\tilde{n}}^{(k)})$ is a quotient of $\bar{\mathcal{U}}_q^{Dr}(X_{\tilde{n}}^{(k)})$.

Since $\bar{\mathcal{U}}_q^{Dr}(X_{\tilde{n}}^{(k)}) \rightarrow \tilde{\mathcal{U}}_q^{Dr}(X_{\tilde{n}}^{(k)})$ is obviously well defined, we just need to prove that this map is surjective, or equivalently that $\tilde{\mathcal{U}}_q^{Dr}(X_{\tilde{n}}^{(k)})$ is generated by (\bar{G}) . To this aim we need some simple remarks.

Remark 3.

In $\tilde{\mathcal{U}}_q^{Dr}$ (hence in \mathcal{U}_q^{Dr}) the following holds:

- i) $\tilde{H}_{i,0}^{\pm} = 1 \quad \forall i \in I_0$;
- ii) $\tilde{H}_{i,\mp r}^{\pm} = 0 \quad \forall i \in I_0, \forall r > 0$;
- iii) $\forall r > 0 \quad \tilde{H}_{i,\pm r}^{\pm} \mp (q_i - q_i^{-1})H_{i,\pm r}$ belongs to the $\mathbb{C}(q)$ -subalgebra generated by $\{H_{i,\pm s} | 0 < s < r\}$; in particular $\{H_{i,\pm s} | (i, s) \in I_{\mathbb{Z}}, 0 < s < r\}$ and $\{\tilde{H}_{i,\pm s}^{\pm} | (i, s) \in I_{\mathbb{Z}}, 0 < s < r\}$ generate the same $\mathbb{C}(q)$ -subalgebra.

Remark 4.

In $\tilde{\mathcal{U}}_q^{Dr}$ (hence in \mathcal{U}_q^{Dr}) we have that for all $i \in I_0$ and for all $r \in \mathbb{Z}_+$

$$\tilde{H}_{i,\pm r}^{\pm} = (q_i - q_i^{-1})k_i^{\mp 1}[X_{i,\pm r}^{\pm}, X_{i,0}^{\mp}].$$

In particular for all $(i, r) \in I_0 \times \mathbb{Z}$ $\tilde{H}_{i,r}^{\pm}$ lies in the subalgebra of $\tilde{\mathcal{U}}_q^{Dr}(X_{\tilde{n}}^{(k)})$ generated by (\bar{G}) .

Consequently (see remark 3) for all $(i, r) \in I_0 \times \mathbb{Z}$ also $H_{i,r}$ lies in the subalgebra of $\tilde{\mathcal{U}}_q^{Dr}(X_{\tilde{n}}^{(k)})$ generated by (\bar{G}) .

Corollary 5.

- i) $\tilde{\mathcal{U}}_q^{Dr}(X_{\tilde{n}}^{(k)})$ and $\mathcal{U}_q^{Dr}(X_{\tilde{n}}^{(k)})$ are generated by (\bar{G}) ;
- ii) $\tilde{\mathcal{U}}_q^{Dr}(X_{\tilde{n}}^{(k)})$ is a quotient of $\bar{\mathcal{U}}_q^{Dr}(X_{\tilde{n}}^{(k)})$.

Notation 6.

We denote by $\tilde{H}_{i,\pm r}^\pm$ also the elements in $\bar{\mathcal{U}}_q^{Dr}$ defined by

$$\tilde{H}_{i,\pm r}^\pm = \begin{cases} (q_i - q_i^{-1})k_i^{-1}[X_{i,r}^+, X_{i,0}^-] & \text{if } r, \pm r > 0 \\ (q_i - q_i^{-1})[X_{i,-r}^-, X_{i,0}^+]k_i & \text{if } r > 0, \pm r < 0 \\ 1 & \text{if } r = 0 \\ 0 & \text{if } r < 0, \end{cases}$$

and by $H_{i,r}$ the elements of $\bar{\mathcal{U}}_q^{Dr}$ defined by

$$\sum_{r \in \mathbb{Z}} \tilde{H}_{i,\pm r}^\pm u^r = \exp \left(\pm (q_i - q_i^{-1}) \sum_{r > 0} H_{i,\pm r} u^r \right);$$

Remark 7.

Relations (ZH) are trivial in $\bar{\mathcal{U}}_q^{Dr}(X_{\tilde{n}}^{(k)})$.

Remark 8.

i) $\bar{\mathcal{U}}_q^{Dr} = \bar{\mathcal{U}}_q^{Dr}(X_{\tilde{n}}^{(k)})$ is Q -graded:

$$\bar{\mathcal{U}}_q^{Dr} = \bigoplus_{\alpha \in Q} \bar{\mathcal{U}}_{q,\alpha}^{Dr},$$

where $C^{\pm 1}, k_i^{\pm 1} \in \bar{\mathcal{U}}_{q,0}^{Dr}, X_{i,r}^\pm \in \bar{\mathcal{U}}_{q,\pm\alpha_i+r\delta}^{Dr} \forall i \in I_0, r \in \mathbb{Z}$, and $\bar{\mathcal{U}}_{q,\alpha}^{Dr} \bar{\mathcal{U}}_{q,\beta}^{Dr} \subseteq \bar{\mathcal{U}}_{q,\alpha+\beta}^{Dr}$.

ii) $\tilde{H}_{i,r}^\pm$ ($r \in \mathbb{Z}$) and $H_{i,r}$ ($r \in \mathbb{Z} \setminus \{0\}$) are homogeneous of degree $r\delta$ for all $i \in I_0$.

iii) Since the relations defining $\tilde{\mathcal{U}}_q^{Dr}$ and \mathcal{U}_q^{Dr} are homogeneous, the Q -gradation of $\bar{\mathcal{U}}_q^{Dr}$ induces Q -gradations on $\tilde{\mathcal{U}}_q^{Dr} = \bigoplus_{\alpha \in Q} \tilde{\mathcal{U}}_{q,\alpha}^{Dr}$ and on $\mathcal{U}_q^{Dr} = \bigoplus_{\alpha \in Q} \mathcal{U}_{q,\alpha}^{Dr}$.

Notation 9.

The $\mathbb{C}(q)$ -algebra $\mathbb{C}(q)[C^{\pm 1}, k_i^{\pm 1} | i \in I_0]$ is Q -graded, with one-dimensional homogeneous components $\mathbb{C}(q)k_\alpha$ ($\alpha \in Q$) where we set

$$k_{m\delta + \sum_{i \in I_0} m_i \alpha_i} = C^m \prod_{i \in I_0} k_i^{m_i} \quad (m, m_i \in \mathbb{Z} \forall i \in I_0).$$

Indeed $\mathbb{C}(q)[C^{\pm 1}, k_i^{\pm 1} | i \in I_0] = \mathbb{C}(q)[Q]$.

Recall that $\mathbb{C}(q)[C^{\pm 1}, k_i^{\pm 1} | i \in I_0]$ naturally maps in $\bar{\mathcal{U}}_{q,0}^{Dr} \subseteq \bar{\mathcal{U}}_q^{Dr}$ (hence in $\tilde{\mathcal{U}}_{q,0}^{Dr} \subseteq \tilde{\mathcal{U}}_q^{Dr}$ and in $\mathcal{U}_{q,0}^{Dr} \subseteq \mathcal{U}_q^{Dr}$).

Remark 10.

i) The relations (CUK) , (CK) and $(KX)^\pm$ are equivalent to a) and b):

- a) the $\mathbb{C}(q)$ -subalgebra generated by $\{C^{\pm 1}, k_i^{\pm 1} | i \in I_0\}$ is a quotient of the ring of Laurent polynomials $\mathbb{C}(q)[k_i^{\pm 1} | i \in I]$ ($C = \prod_{i \in I} k_i^{r_i}$);
- b) for all $\alpha, \beta \in Q$ and for all x of degree β we have $k_\alpha x = q^{(\alpha|\beta)} x k_\alpha$.
- ii) Relations (KH) depend on relations (CUK) , (CK) and $(KX)^\pm$, and in particular are trivial in $\tilde{\mathcal{U}}_q^{Dr}(X_{\tilde{n}}^{(k)})$.

Definition 11.

We denote by $\mathcal{F}_q^+ = \mathcal{F}_q^+(X_{\tilde{n}}^{(k)})$ and $\mathcal{F}_q^- = \mathcal{F}_q^-(X_{\tilde{n}}^{(k)})$ the $\mathbb{C}(q)$ -algebras generated respectively by

$$(G^+) \quad X_{i,r}^+ \quad ((i, r) \in I_0 \times \mathbb{Z})$$

and

$$(G^-) \quad X_{i,r}^- \quad ((i, r) \in I_0 \times \mathbb{Z})$$

with relations respectively (ZX^+) and (ZX^-) .

Remark 12.

$\mathcal{F}_q^+(X_{\tilde{n}}^{(k)})$ and $\mathcal{F}_q^-(X_{\tilde{n}}^{(k)})$ are the free $\mathbb{C}(q)$ -algebras generated respectively by

$$(G'^+) \quad X_{i,r}^+ \quad ((i, r) \in I_{\mathbb{Z}})$$

and

$$(G'^-) \quad X_{i,r}^- \quad ((i, r) \in I_{\mathbb{Z}}).$$

Notation 13.

\mathcal{F}_q^+ and \mathcal{F}_q^- naturally embed in $\bar{\mathcal{U}}_q^{Dr}$, hence they map in $\tilde{\mathcal{U}}_q^{Dr}$ and in \mathcal{U}_q^{Dr} ; their images in $\tilde{\mathcal{U}}_q^{Dr}$ are denoted respectively by $\tilde{\mathcal{U}}_q^{Dr,+}$ and $\tilde{\mathcal{U}}_q^{Dr,-}$, and their images in \mathcal{U}_q^{Dr} are denoted respectively by $\mathcal{U}_q^{Dr,+}$ and $\mathcal{U}_q^{Dr,-}$.

Remark 14.

- i) As subalgebras of $\bar{\mathcal{U}}_q^{Dr}$, \mathcal{F}_q^+ inherits a $(Q_{0,+} \oplus \mathbb{Z}\delta)$ -gradation and \mathcal{F}_q^- inherits a $(-Q_{0,+} \oplus \mathbb{Z}\delta)$ -gradation;
- ii) more precisely we have that

$$\mathcal{F}_q^\pm \subseteq \mathbb{C}(q) \oplus \bigoplus_{\substack{\alpha \in Q_{0,+}, \alpha \neq 0 \\ m \in \mathbb{Z}}} \bar{\mathcal{U}}_{q,\pm\alpha+m\delta}^{Dr}$$

and similarly

$$\tilde{\mathcal{U}}_q^{Dr, \pm} \subseteq \mathbb{C}(q) \oplus \bigoplus_{\substack{\alpha \in Q_{0,+}, \alpha \neq 0 \\ m \in \mathbb{Z}}} \tilde{\mathcal{U}}_{q, \pm\alpha+m\delta}^{Dr}$$

and

$$\mathcal{U}_q^{Dr, \pm} \subseteq \mathbb{C}(q) \oplus \bigoplus_{\substack{\alpha \in Q_{0,+}, \alpha \neq 0 \\ m \in \mathbb{Z}}} \mathcal{U}_{q, \pm\alpha+m\delta}^{Dr}.$$

The last part of this section is devoted to the definition of automorphisms and antiautomorphisms of the algebras just introduced, which make evident some symmetries in the generators and relations of \mathcal{U}_q^{Dr} . Thanks to these structures the study of the apparently very complicated relations defining \mathcal{U}_q^{Dr} will be strongly simplified in sections §7, §9 and following.

The next definitions depend on the choice of an automorphism η of \mathbb{C} . A short discussion about the choice of η is outlined in remark 17.

Definition 15.

Let us introduce the following homomorphisms and antihomomorphisms:

i) $\bar{\Omega} : \bar{\mathcal{U}}_q^{Dr} \rightarrow \bar{\mathcal{U}}_q^{Dr}$ is the anti-homomorphism defined on the generators by

$$\bar{\Omega}|_{\mathbb{C}} = \eta, \quad q \mapsto q^{-1}, \quad C^{\pm 1} \mapsto C^{\mp 1}, \quad k_i^{\pm 1} \mapsto k_i^{\mp 1}, \quad X_{i,r}^{\pm} \mapsto X_{i,-r}^{\mp}.$$

ii) $\Theta_{\mathcal{F}}^+ : \mathcal{F}_q^+ \rightarrow \mathcal{F}_q^+$ and $\Theta_{\mathcal{F}}^- : \mathcal{F}_q^- \rightarrow \mathcal{F}_q^-$ are the homomorphisms defined on the generators respectively by

$$\Theta_{\mathcal{F}}^+ : \quad \Theta_{\mathcal{F}}^+|_{\mathbb{C}} = \eta, \quad q \mapsto q^{-1}, \quad X_{i,r}^+ \mapsto X_{i,-r}^+$$

and

$$\Theta_{\mathcal{F}}^- : \quad \Theta_{\mathcal{F}}^-|_{\mathbb{C}} = \eta, \quad q \mapsto q^{-1}, \quad X_{i,r}^- \mapsto X_{i,-r}^-.$$

iii) $\bar{\Theta} : \bar{\mathcal{U}}_q^{Dr} \rightarrow \bar{\mathcal{U}}_q^{Dr}$ is the homomorphism defined on the generators by

$$\begin{aligned} \bar{\Theta}|_{\mathbb{C}} &= \eta, \quad q \mapsto q^{-1}, \quad C^{\pm 1} \mapsto C^{\pm 1}, \quad k_i^{\pm 1} \mapsto k_i^{\mp 1}, \\ X_{i,r}^+ &\mapsto -X_{i,-r}^+ k_i C^{-r}, \quad X_{i,r}^- \mapsto -k_i^{-1} C^{-r} X_{i,-r}^-. \end{aligned}$$

iv) For all $i \in I_0$ $\bar{t}_i : \bar{\mathcal{U}}_q^{Dr} \rightarrow \bar{\mathcal{U}}_q^{Dr}$ is the $\mathbb{C}(q)$ -homomorphism defined on the generators by

$$C^{\pm 1} \mapsto C^{\pm 1}, \quad k_j^{\pm 1} \mapsto (k_j C^{-\delta_{ij} \tilde{d}_i})^{\pm 1}, \quad X_{j,r}^{\pm} \mapsto X_{j, r \mp \delta_{ij} \tilde{d}_i}^{\pm}.$$

v) For $i \in I_0$ let

$$\bar{\phi}_i : \begin{cases} \bar{\mathcal{U}}_q^{Dr}(A_1^{(1)}) \rightarrow \bar{\mathcal{U}}_q^{Dr}(X_{\tilde{n}}^{(k)}) & \text{if } (X_{\tilde{n}}^{(k)}, i) \neq (A_{2n}^{(2)}, 1) \\ \bar{\mathcal{U}}_q^{Dr}(A_2^{(2)}) \rightarrow \bar{\mathcal{U}}_q^{Dr}(X_{\tilde{n}}^{(k)}) & \text{if } (X_{\tilde{n}}^{(k)}, i) = (A_{2n}^{(2)}, 1) \end{cases}$$

be the \mathbb{C} -homomorphisms defined on the generators as follows:

$$q \mapsto q_i, \quad C^{\pm 1} \mapsto C^{\pm \tilde{d}_i}, \quad k^{\pm 1} \mapsto k_i^{\pm 1}, \quad X_r^{\pm} \mapsto X_{i, \tilde{d}_i r}^{\pm}.$$

Remark 16.

It is immediate to notice that:

- i) $\bar{\Omega}$, $\Theta_{\mathcal{F}}^{\pm}$, $\bar{\Theta}$, \bar{t}_i and $\bar{\phi}_i$ are all well-defined;
 - ii) $\bar{\Omega}(\mathcal{F}_q^{\pm}) = \mathcal{F}_q^{\mp}$;
 - iii) $\bar{\Omega}$ and $\bar{\Theta}$ are involutions of $\bar{\mathcal{U}}_q^{Dr}$, if η is an involution of \mathbb{C} ;
 - iv) the \bar{t}_i 's are automorphisms of $\bar{\mathcal{U}}_q^{Dr}$ (of infinite order) for all $i \in I_0$;
- more precisely $\langle \bar{t}_i | i \in I_0 \rangle \cong Z^{I_0}$;
- v) the following commutation properties hold:

$$\bar{\Theta}\bar{\Omega} = \bar{\Omega}\bar{\Theta}, \quad \bar{t}_i\bar{\Omega} = \bar{\Omega}\bar{t}_i, \quad \bar{t}_i\bar{\Theta} = \bar{\Theta}\bar{t}_i^{-1} \quad \text{and} \quad \bar{t}_i\bar{t}_j = \bar{t}_j\bar{t}_i \quad \forall i, j \in I_0$$

as maps of $\bar{\mathcal{U}}_q^{Dr}(X_{\tilde{n}}^{(k)})$ into itself; moreover, $\forall i \in I_0$,

$$\bar{\Omega}\bar{\phi}_i = \bar{\phi}_i\bar{\Omega}, \quad \bar{\Theta}\bar{\phi}_i = \bar{\phi}_i\bar{\Theta}, \quad \bar{t}_i\bar{\phi}_i = \bar{\phi}_i\bar{t}_i, \quad \bar{t}_j\bar{\phi}_i = \bar{\phi}_i \quad \forall j \in I_0 \setminus \{i\}$$

as maps from $\bar{\mathcal{U}}_q^{Dr}(A_1^{(1)})$ to $\bar{\mathcal{U}}_q^{Dr}(X_{\tilde{n}}^{(k)})$ if $(X_{\tilde{n}}^{(k)}, i) \neq (A_{2n}^{(2)}, 1)$, from $\bar{\mathcal{U}}_q^{Dr}(A_2^{(2)})$ to $\bar{\mathcal{U}}_q^{Dr}(X_{\tilde{n}}^{(k)})$ if $(X_{\tilde{n}}^{(k)}, i) = (A_{2n}^{(2)}, 1)$;

- vi) for all $\alpha = \beta + m\delta \in Q$ with $\beta \in Q_0$, $m \in \mathbb{Z}$ we have:

$$\bar{\Omega}(k_{\alpha}) = k_{-\alpha}, \quad \bar{\Theta}(k_{\beta+m\delta}) = k_{-\beta+m\delta}, \quad \bar{t}_i(k_{\alpha}) = k_{\lambda_i(\alpha)}$$

and

$$\bar{\Omega}(\bar{\mathcal{U}}_{q, \alpha}^{Dr}) = \bar{\mathcal{U}}_{q, -\alpha}^{Dr}, \quad \bar{\Theta}(\bar{\mathcal{U}}_{q, \beta+m\delta}^{Dr}) = \bar{\mathcal{U}}_{q, \beta-m\delta}^{Dr}, \quad \bar{t}_i(\bar{\mathcal{U}}_{q, \alpha}^{Dr}) = \bar{\mathcal{U}}_{q, \lambda_i(\alpha)}^{Dr};$$

moreover for all $m_1, m \in \mathbb{Z}$ and for all $i \in I_0$

$$\bar{\phi}_i(k_{m_1\alpha_1+m\delta}) = k_{m_1\alpha_i+\tilde{d}_im\delta} \quad \text{and} \quad \bar{\phi}_i(\mathcal{U}_{q, m_1\alpha_1+m\delta}^{Dr}(A_*^{(*)})) \subseteq \mathcal{U}_{q, m_1\alpha_i+\tilde{d}_im\delta}^{Dr}(X_{\tilde{n}}^{(k)});$$

- vii) on the elements $H_{i,r}$ and $\tilde{H}_{i,r}^{\pm}$ we have:

$$\bar{\Omega}(\tilde{H}_{i,r}^{\pm}) = \tilde{H}_{i,-r}^{\mp}, \quad \bar{\Omega}(H_{i,r}) = H_{i,-r}$$

and

$$\bar{\phi}_i(\tilde{H}_{1,r}^{\pm}) = \tilde{H}_{i, \tilde{d}_i r}^{\pm}, \quad \bar{\phi}_i(H_{1,r}) = H_{i, \tilde{d}_i r} \quad \forall i \in I_0.$$

Remark 17.

For the purpose of the present paper, the definition of $\bar{\Omega}$, $\Theta_{\mathcal{F}}^{\pm}$, $\bar{\Theta}$ given in definition 15 could be simplified by requiring these maps to be \mathbb{C} -linear (that is $\eta = id_{\mathbb{C}}$). But the choice of a non trivial automorphism η of \mathbb{C} becomes sometimes necessary, as when specializing q at a complex value $\epsilon \neq \pm 1$: indeed a homomorphism defined over $\mathbb{C}(q)$ (and mapping q to q^{-1}) induces a homomorphism on the specialization at ϵ if and only if the ideal $(q - \epsilon)$ is stable; if, for example, ϵ is a root of 1, this could

be obtained by choosing $\eta(z) = \bar{z} \forall z \in \mathbb{C}$, that is by requiring the homomorphism to be \mathbb{C} -anti-linear. For this reason, from now on we suppose η to be the conjugation on \mathbb{C} , that is $\bar{\Omega}$, $\Theta_{\mathcal{F}}^{\pm}$, $\bar{\Theta}$ to be \mathbb{C} -anti-linear (see definitions §8.2 and §8.4, and compare also with definition §3.3).

Of course one needs to pay more attention and eventually to choose a different η when interested in specializing at complex values ϵ such that $|\epsilon| \neq 1$.

Our goal is of course to show that $\bar{\Omega}$, $\bar{\Theta}$, \bar{t}_i and $\bar{\phi}_i$ induce Ω , Θ , t_i and ϕ_i on \mathcal{U}_q^{Dr} . This is indeed very easy to show, but we take this occasion to simplify the relations that we have to handle with, passing through $\tilde{\mathcal{U}}_q^{Dr}$ for two reasons: underlying the first redundances of the relations (see corollary §7.17); discussing separately the relations $(XD)^+-(S3)^+$ whose first simplification can be dealt with simultaneously as examples of a general case (see section §9).

§7. The ALGEBRA $\tilde{\mathcal{U}}_q^{Dr}$.

The algebra $\tilde{\mathcal{U}}_q^{Dr}$ and its structures, to which this section is devoted, play a fundamental role in the study and simplification of relations $(XD^{\pm})-(S3^{\pm})$. In particular the relations are analyzed underlining their consequences on the (anti)automorphisms $\tilde{\Omega}$, $\tilde{\Theta}$ and \tilde{t}_i ($i \in I_0$); relations (HX^{\pm}) and (HH) are proved to be redundant; and much smaller sets of generators are provided.

Remark 1.

Remarks §6.10,i) and §6.16,vi) imply immediately that $\bar{\Omega}$, $\bar{\Theta}$, \bar{t}_i and $\bar{\phi}_i$ preserve relations $(KX)^{\pm}$.

Remark 2.

For all $i \in I_0$ $\bar{\phi}_i$ obviously induces

$$\tilde{\phi}_i : \begin{cases} \tilde{\mathcal{U}}_q^{Dr}(A_1^{(1)}) \rightarrow \tilde{\mathcal{U}}_q^{Dr}(X_{\tilde{n}}^{(k)}) & \text{if } (X_{\tilde{n}}^{(k)}, i) \neq (A_{2n}^{(2)}, 1) \\ \tilde{\mathcal{U}}_q^{Dr}(A_2^{(2)}) \rightarrow \tilde{\mathcal{U}}_q^{Dr}(X_{\tilde{n}}^{(k)}) & \text{if } (X_{\tilde{n}}^{(k)}, i) = (A_{2n}^{(2)}, 1) \end{cases}$$

and

$$\phi_i : \begin{cases} \mathcal{U}_q^{Dr}(A_1^{(1)}) \rightarrow \mathcal{U}_q^{Dr}(X_{\tilde{n}}^{(k)}) & \text{if } (X_{\tilde{n}}^{(k)}, i) \neq (A_{2n}^{(2)}, 1) \\ \mathcal{U}_q^{Dr}(A_2^{(2)}) \rightarrow \mathcal{U}_q^{Dr}(X_{\tilde{n}}^{(k)}) & \text{if } (X_{\tilde{n}}^{(k)}, i) = (A_{2n}^{(2)}, 1) \end{cases}.$$

Remark 3.

- i) $\bar{\Omega}(\mathcal{I}^+(HXL)) = \mathcal{I}^-(HXL)$ and $\bar{\Omega}(\mathcal{I}^+(HX)) = \mathcal{I}^-(HX)$;
- ii) $\bar{\Omega}$ preserves relations $(HXL)^\pm$ and relations $(HX)^\pm$.

Notation 4.

Define relations (XXD) (XXE) , (XXH^+) and (XXH^-) by:

$$(XXD) \quad [X_{i,r}^+, X_{j,s}^-] = 0 \quad ((i, r), (j, s) \in I_{\mathbb{Z}}, i \neq j),$$

$$(XXE) \quad [X_{i,r}^+, X_{i,-r}^-] = \frac{C^r k_i - C^{-r} k_i^{-1}}{q_i - q_i^{-1}} \quad ((i, r) \in I_{\mathbb{Z}}),$$

$$(XXH^+) \quad [X_{i,r}^+, X_{i,s}^-] = \frac{C^{-s} k_i \tilde{H}_{i,r+s}^+}{q_i - q_i^{-1}} \quad ((i, r), (i, s) \in I_{\mathbb{Z}}, r + s > 0),$$

$$(XXH^-) \quad [X_{i,r}^+, X_{i,s}^-] = -\frac{C^{-r} \tilde{H}_{i,r+s}^- k_i^{-1}}{q_i - q_i^{-1}} \quad ((i, r), (i, s) \in I_{\mathbb{Z}}, r + s < 0),$$

Remark 5.

- i) $\mathcal{I}(XX) = \mathcal{I}(XXD, XXE, XXH)$;
- ii) $\bar{\Omega}(\mathcal{I}(XXD)) = \mathcal{I}(XXD)$ and $\bar{\Omega}(\mathcal{I}(XXE)) = \mathcal{I}(XXE)$;
- iii) $\bar{\Omega}(\mathcal{I}(XXH^+)) = \mathcal{I}(XXH^+)$;
- iv) $\bar{\Omega}$ preserves relations (XX) .

Corollary 6.

$\bar{\Omega}$ induces $\tilde{\Omega} : \tilde{\mathcal{U}}_q^{Dr} \rightarrow \tilde{\mathcal{U}}_q^{Dr}$.

Remark 7.

- i) $\bar{t}_i(\mathcal{I}(XXD)) = \mathcal{I}(XXD)$ and $\bar{t}_i(\mathcal{I}(XXE)) = \mathcal{I}(XXE) \forall i \in I_0$;
- ii) $\mathcal{I}(XXD, XXE)$ is the $\bar{t}_i^{\pm 1}$ -stable ideal ($\forall i \in I_0$) generated by

$$\left\{ [X_{i,0}^+, X_{j,0}^-] - \delta_{ij} \frac{k_i - k_i^{-1}}{q_i - q_i^{-1}} | i, j \in I_0 \right\}.$$

We want to show now that for all $i \in I_0$ \bar{t}_i induces $\tilde{t}_i : \tilde{\mathcal{U}}_q^{Dr} \rightarrow \tilde{\mathcal{U}}_q^{Dr}$. Since \bar{t}_i commutes with $\bar{\Omega}$ remarks 1, 3,i), 5,i) and iii) and 7,i) imply that it is enough to concentrate on $\mathcal{I}(HXL^+)$, $\mathcal{I}(XXH^+)$.

Remark 8.

- i) Remark that if $r + s > 0$

$$(q_i - q_i^{-1}) C^s k_i^{-1} [X_{i,r}^+, X_{i,s}^-] = \bar{t}_i^{s/\bar{d}_i} ((q_i - q_i^{-1}) k_i^{-1} [X_{i,r+s}^+, X_{i,0}^-]) = \bar{t}_i^{s/\bar{d}_i} (\tilde{H}_{i,r+s}^+),$$

so that relations $(XXH)^+$ are equivalent to

$$\bar{t}_i^s(\tilde{H}_{i,r}^+) = \tilde{H}_{i,r}^+ \quad \forall i \in I_0, \quad r > 0, \quad s \in \mathbb{Z};$$

ii) $\bar{t}_i^{\pm 1}(\mathcal{I}(XXH^+)) = \mathcal{I}(XXH^+) \quad \forall i \in I_0$;

iii) $\mathcal{I}^+(XXH)$ is the $\bar{t}_i^{\pm 1}$ -stable ideal ($\forall i \in I_0$) generated by

$$\{\bar{t}_i(\tilde{H}_{i,r}^+) - \tilde{H}_{i,r}^+ | i \in I_0, \quad r > 0\}.$$

Remark 9.

Remark that for all $(i, r) \in I_{\mathbb{Z}} \setminus (I_0 \times \{0\})$, $(j, s) \in I_{\mathbb{Z}}$ and $h \in I_0$

$$\bar{t}_h^{\pm 1}([H_{i,r}, X_{j,s}^+] - b_{ijr} C^{\frac{r-|r|}{2}} X_{j,r+s}^+) = [\bar{t}_h^{\pm 1}(H_{i,r}), X_{j,s \mp \delta_{jh} \tilde{d}_j}^+] - b_{ijr} C^{\frac{r-|r|}{2}} X_{j,r+s \mp \delta_{jh} \tilde{d}_j}^+.$$

Then, thanks to remark 8 and to the definition of \bar{t}_i (see definition §6.15), we have that:

i) $\bar{t}_i^{\pm 1}(\mathcal{I}^+(HX)) \subseteq \mathcal{I}^+(XXH, HX)$;

ii) $\bar{t}_i^{\pm 1}(\mathcal{I}^+(HXL)) \subseteq \mathcal{I}^+(XXH, HXL)$;

iii) $\mathcal{I}^+(XXH, HXL)$ is the $\bar{t}_i^{\pm 1}$ -stable ideal ($\forall i \in I_0$) generated by

$$\{\bar{t}_i(\tilde{H}_{i,r}^+) - \tilde{H}_{i,r}^+, [H_{i,s}, X_{j,0}^+] - b_{ijs} C^{\frac{s-|s|}{2}} X_{j,s}^+ | i \in I_0, \quad r > 0, \quad \tilde{d}_i \leq |s| \leq \tilde{d}_{ij}\}.$$

Corollary 10.

i) For all $i \in I_0$ \bar{t}_i induces $\tilde{t}_i : \tilde{\mathcal{U}}_q^{Dr} \rightarrow \tilde{\mathcal{U}}_q^{Dr}$;

ii) For all $i, j \in I_0$ $\tilde{t}_i(\tilde{H}_{j,r}^+) = \tilde{H}_{j,r}^+ \quad \forall r \in \mathbb{Z}$ and $\tilde{t}_i(H_{j,r}) = H_{j,r} \quad \forall r \in \mathbb{Z} \setminus \{0\}$.

We come now to $\bar{\Theta}$ recalling that $\bar{\Theta}\bar{\Omega} = \bar{\Omega}\bar{\Theta}$ and $\bar{\Theta}\bar{t}_i^{\pm 1} = \bar{t}_i^{\mp 1}\bar{\Theta}$ for all $i \in I_0$.

Remark 11.

Notice that $[X_{i,0}^+, X_{j,0}^-] - \delta_{ij} \frac{k_i - k_i^{-1}}{q_i - q_i^{-1}}$ is fixed by $\bar{\Theta}$; hence, thanks to remark 7,ii), $\mathcal{I}(XXD, XXE)$ is $\bar{\Theta}$ -stable.

Remark 12.

i) For all $i \in I_0$ and for all $r > 0$

$$\bar{\Theta}(\tilde{H}_{i,r}^+) = \bar{t}_i^{\frac{r}{\tilde{d}_i}}(\tilde{H}_{i,-r}^-) + (q_i - q_i^{-1})[X_{i,-r}^+ X_{i,0}^-, k_i] C^{-r};$$

ii) for all $i \in I_0$ and for all $r > 0$ $\bar{\Theta}(\tilde{H}_{i,\pm r}^{\pm}) - \tilde{H}_{i,\mp r}^{\mp}$ and $\bar{\Theta}(H_{i,\pm r}) - H_{i,\mp r}$ lie in $\mathcal{I}(KX, XXH^{\mp})$;

iii) for all $i \in I_0$ and for all $r > 0$

$$\bar{\Theta}(\bar{t}_i(\tilde{H}_{i,r}^+) - \tilde{H}_{i,r}^+) = \bar{t}_i^{-1}(\bar{\Theta}(\tilde{H}_{i,r}^+)) - \bar{\Theta}(\tilde{H}_{i,r}^+) \in \mathcal{I}(KX, XXH^-);$$

iv) for all $i, j \in I_0$, $\tilde{d}_i \leq |r| \leq \tilde{d}_{ij}$, $s \in \mathbb{Z}$

$$\begin{aligned} & \bar{\Theta}([H_{i,r}, X_{j,s}^+] - b_{ijr} C^{\frac{r-|r|}{2}} X_{j,r+s}^+) = \\ & = -[\bar{\Theta}(H_{i,r}), X_{j,-s}^+ k_j C^{-s}] + b_{ijr} C^{\frac{r-|r|}{2}} X_{j,-(r+s)}^+ k_j C^{-(r+s)} = \\ & = -([\bar{\Theta}(H_{i,r}), X_{j,-s}^+ k_j] k_j^{-1} - b_{ijr} C^{\frac{-r-|r|}{2}} X_{j,-(r+s)}^+) k_j C^{-s} \end{aligned}$$

belongs to $\mathcal{I}(KX, XXH, HXL^+)$;

Then:

v) $\bar{\Theta}(\mathcal{I}(XXH^+)) \subseteq \mathcal{I}(KX, XXH^-)$;

vi) $\bar{\Theta}(\mathcal{I}(HXL^+)) \subseteq \mathcal{I}(KX, XXH, HXL^+)$;

vii) $\mathcal{I}(KX, XXH)$ and $\mathcal{I}(KX, XXH, HXL^\pm)$ are $\bar{\Theta}$ -stable.

Corollary 13.

i) $\bar{\Theta}$ induces $\tilde{\Theta} : \tilde{\mathcal{U}}_q^{Dr} \rightarrow \tilde{\mathcal{U}}_q^{Dr}$;

ii) For all $i \in I_0$ $\tilde{\Theta}(\tilde{H}_{i,r}^+) = \tilde{H}_{i,-r}^- \forall r \in \mathbb{Z}$ and $\tilde{\Theta}(H_{i,r}) = H_{i,-r} \forall r \in \mathbb{Z} \setminus \{0\}$.

Remark 14.

i) Let $f : Q_{0,+} \rightarrow \mathbb{Z}$ be defined by:

$$f(0) = 0, \quad f(\alpha + \alpha_i) = f(\alpha) + (\alpha|\alpha_i) \quad \forall \alpha \in Q_{0,+}, \quad i \in I_0;$$

notice that f is well defined, because $(\alpha|\alpha_i) + (\alpha + \alpha_i|\alpha_j) = (\alpha|\alpha_j) + (\alpha + \alpha_j|\alpha_i)$.

ii) $\forall X^+ \in \mathcal{F}_{q,\alpha+m\delta}^+$ and $\forall X^- \in \mathcal{F}_{q,-\alpha+m\delta}^-$ (where $\alpha \in Q_{0,+}$, $m \in \mathbb{Z}$) we have that in $\tilde{\mathcal{U}}_q^{Dr}$

$$\tilde{\Theta}\pi^+(X^+) = (-1)^h q^{f(\alpha)} \pi^+ \Theta_{\mathcal{F}}^+(X^+) k_\alpha C^{-m}$$

and

$$\tilde{\Theta}\pi^-(X^-) = (-1)^h q^{-f(\alpha)} C^{-m} k_{-\alpha} \pi^- \Theta_{\mathcal{F}}^-(X^-),$$

where $\pi^\pm : \mathcal{F}_q^\pm \rightarrow \tilde{\mathcal{U}}_q^{Dr}$ is the restriction to \mathcal{F}_q^\pm of the natural projection $\bar{\mathcal{U}}_q^{Dr} \rightarrow \tilde{\mathcal{U}}_q^{Dr}$ and $h = \sum_{i \in I_0} m_i$ if $\alpha = \sum_{i \in I_0} m_i \alpha_i$.

iii) In particular $\tilde{\Theta}\pi^\pm(X^\pm)$ and $\pi^\pm \Theta_{\mathcal{F}}^\pm(X^\pm)$ are equal up to invertible elements of $\tilde{\mathcal{U}}_q^{Dr}$.

We present now some more remarks about generators and relations of $\tilde{\mathcal{U}}_q^{Dr}$.

For the next proposition see the analogous results for \mathcal{U}_q^{DJ} , in [Be] and [Da].

Proposition 15.

In $\tilde{\mathcal{U}}_q^{Dr}$ we have $\mathcal{I}(HX^\pm) \subseteq \mathcal{I}(XD^\pm, X1^\pm, X2^\pm)$.

Proof: In order to avoid repetitive computations we use the behaviour of relations (XD^\pm) , $(X1^\pm)$ and $(X2^\pm)$ under the action of $\tilde{\Omega}$, $\tilde{\Theta}$ and \tilde{t}_i ($i \in I_0$), which is an independent result proved in remarks §9.8 and §9.20: here it allows us to reduce to the study of $[H_{i,r}, X_{j,0}^+]$ with $r > 0$. Indeed:

$$\tilde{t}_j^s([H_{i,r}, X_{j,0}^+]) = [H_{i,r}, X_{j,-\tilde{d}_{js}}^+] \quad \text{and} \quad \tilde{t}_j^s(X_{j,r}^+) = X_{j,r-\tilde{d}_{js}}^+$$

$$(\text{moreover } [H_{i,r}, X_{j,s}^+] = 0 \quad \text{and} \quad b_{ijr}X_{j,r+s}^+ = 0 \quad \text{if } \tilde{d}_j \nmid s),$$

$$\tilde{\Theta}([H_{i,r}, X_{j,s}^+]) = -[H_{i,-r}, X_{j,-s}^+]k_jC^{-s} \quad \text{and} \quad \tilde{\Theta}(X_{j,r+s}^+) = -X_{j,-r-s}^+k_jC^{-r-s},$$

$$\tilde{\Omega}([H_{i,r}, X_{j,s}^+]) = -[H_{i,-r}, X_{j,-s}^-] \quad \text{and} \quad \tilde{\Omega}(C^{\frac{r-|r|}{2}}X_{j,r+s}^+) = C^{\frac{-r+|r|}{2}}X_{j,-r-s}^-,$$

and of course $b_{ijr} = \tilde{t}_j(b_{ijr}) = \tilde{\Theta}(b_{ijr}) = \tilde{\Omega}(b_{ijr}) = b_{ij-r}$;

Given an element $x \in \tilde{\mathcal{U}}_q^{Dr}$ define the operators ${}_l x$ and ${}_r x$ on $\tilde{\mathcal{U}}_q^{Dr}$ respectively as the left and right multiplication by x ; if we have elements $x_s \in \tilde{\mathcal{U}}_q^{Dr}$ ($s \in \mathbb{N}$) set ${}_l x(u) = \sum_{s \in \mathbb{N}} {}_l x_s u^s$ and ${}_r x(u) = \sum_{s \in \mathbb{N}} {}_r x_s u^s$; notice that if $f : \tilde{\mathcal{U}}_q^{Dr} \rightarrow \tilde{\mathcal{U}}_q^{Dr}$ is such that $f(x_s) = x_s$ for all $s \in \mathbb{N}$ then ${}_l x(u)$ and ${}_r x(u)$ commute with f .

Let $i, j \in I_0$: we want to study $({}_l \tilde{H}_i^+(u) - {}_r \tilde{H}_i^+(u))(X_{j,s}^+)$ and deduce from it $({}_l H_i(u) - {}_r H_i(u))(X_{j,s}^+)$ (setting $H_{i,0} = 0$). To this aim remark that

$${}_l \tilde{H}_i^+(u) = \exp((q_i - q_i^{-1}){}_l H_i(u)), \quad {}_r \tilde{H}_i^+(u) = \exp((q_i - q_i^{-1}){}_r H_i(u))$$

and both commute with \tilde{t}_j .

The next computations are performed in $\tilde{\mathcal{U}}_q^{Dr}/\mathcal{I}(XD^+, X1^+, X2^+)$.

Remarking that ($r > \tilde{d}_{ij}$)

$$\begin{aligned} [\tilde{H}_{i,r}^+, X_{j,0}^+] &= (q_i - q_i^{-1})k_i^{-1}[[X_{i,r}^+, X_{i,0}^-], X_{j,0}^+]_{q_i^{a_{ij}}} = \\ &= (q_i - q_i^{-1})k_i^{-1}([X_{i,r}^+, [X_{i,0}^-, X_{j,0}^+]]_{q_i^{a_{ij}}} - [X_{i,0}^-, [X_{i,r}^+, X_{j,0}^+]]_{q_i^{a_{ij}}}) = \\ &= (q_i - q_i^{-1})k_i^{-1}(q_i^{a_{ij}}[\delta_{ij}\frac{k_i - k_i^{-1}}{q_i - q_i^{-1}}, X_{i,r}^+]_{q_i^{-a_{ij}}} + [[X_{i,r}^+, X_{j,0}^+]_{q_i^{a_{ij}}}, X_{i,0}^-]) = \\ &= (q_i - q_i^{-1})k_i^{-1}(\delta_{ij}[2]_{q_i}k_iX_{i,r}^+ + [[X_{i,r}^+, X_{j,0}^+]_{q_i^{a_{ij}}}, X_{i,0}^-]), \end{aligned}$$

let us distinguish two cases:

i) $(X_{\tilde{n}}^{(k)}, i, j) \neq (A_{2n}^{(2)}, 1, 1)$: then, thanks to (XD^+) , $(X1^+)$ and (HXL^+) , we have

$$\begin{aligned} [\tilde{H}_{i,r}^+, X_{j,0}^+] &= (q_i - q_i^{-1})k_i^{-1}(\delta_{ij}[2]_{q_i}k_iX_{i,r}^+ - [[X_{j,\tilde{d}_{ij}}^+, X_{i,r-\tilde{d}_{ij}}^+]_{q_i^{a_{ij}}}, X_{i,0}^-]) = \\ &= (q_i - q_i^{-1})k_i^{-1}(\delta_{ij}[2]_{q_i}k_iX_{i,r}^+ - [X_{j,\tilde{d}_{ij}}^+, [X_{i,r-\tilde{d}_{ij}}^+, X_{i,0}^-]]_{q_i^{a_{ij}}} + \end{aligned}$$

$$\begin{aligned}
 & + q_i^{a_{ij}} [X_{i,r-\tilde{d}_{ij}}^+, [X_{j,\tilde{d}_{ij}}^+, X_{i,0}^-]_{q_i^{-a_{ij}}}] = \\
 & = (q_i - q_i^{-1}) \delta_{ij} ([2]_{q_i} X_{i,r}^+ - \frac{1}{q_i - q_i^{-1}} [\tilde{H}_{i,\tilde{d}_i}^+, X_{i,r-\tilde{d}_i}^+]) + \\
 & + (q_i - q_i^{-1}) q_i^{a_{ij}} k_i^{-1} [[X_{i,r-\tilde{d}_{ij}}^+, X_{i,0}^-], X_{j,\tilde{d}_{ij}}^+]_{q_i^{-a_{ij}}} = \\
 & = (q_i - q_i^{-1}) \delta_{ij} ([2]_{q_i} X_{i,r}^+ - [H_{i,\tilde{d}_i}, X_{i,r-\tilde{d}_i}^+]) + q_i^{a_{ij}} [\tilde{H}_{i,r-\tilde{d}_{ij}}^+, X_{j,\tilde{d}_{ij}}^+]_{q_i^{-2a_{ij}}} = \\
 & = q_i^{a_{ij}} \tilde{H}_{i,r-\tilde{d}_{ij}}^+ X_{j,\tilde{d}_{ij}}^+ - q_i^{-a_{ij}} X_{j,\tilde{d}_{ij}}^+ \tilde{H}_{i,r-\tilde{d}_{ij}}^+;
 \end{aligned}$$

hence, using again (HXL^+) ,

$$(\iota \tilde{H}_i^+(u) - {}_r \tilde{H}_i^+(u))(X_{j,0}^+) = (q_i^{a_{ij}} \iota \tilde{H}_i^+(u) - q_i^{-a_{ij}} {}_r \tilde{H}_i^+(u)) \tilde{t}_j^{\frac{\tilde{d}_{ij}}{d_j}} u^{\tilde{d}_{ij}} (X_{j,0}^+),$$

or equivalently

$$\iota \tilde{H}_i^+(u) (1 - q_i^{a_{ij}} \tilde{t}_j^{\frac{\tilde{d}_{ij}}{d_j}} u^{\tilde{d}_{ij}}) (X_{j,0}^+) = {}_r \tilde{H}_i^+(u) (1 - q_i^{-a_{ij}} \tilde{t}_j^{\frac{\tilde{d}_{ij}}{d_j}} u^{\tilde{d}_{ij}}) (X_{j,0}^+);$$

from this we get

$$\begin{aligned}
 & (q_i - q_i^{-1}) (\iota H_i(u) - {}_r H_i(u)) (X_{j,0}^+) = \\
 & = \log(1 - q_i^{-a_{ij}} \tilde{t}_j^{\frac{\tilde{d}_{ij}}{d_j}} u^{\tilde{d}_{ij}}) - \log(1 - q_i^{a_{ij}} \tilde{t}_j^{\frac{\tilde{d}_{ij}}{d_j}} u^{\tilde{d}_{ij}}),
 \end{aligned}$$

that is

$$[H_{i,r}, X_{j,0}^+] = \begin{cases} 0 & \text{if } \tilde{d}_{ij} \nmid r \\ \frac{q_i^{\frac{ra_{ij}}{d_{ij}}} - q_i^{-\frac{ra_{ij}}{d_{ij}}}}{\frac{r}{d_{ij}}(q_i - q_i^{-1})} X_{j,r}^+ = b_{ijr} X_{j,r}^+ & \text{otherwise.} \end{cases}$$

ii) $(X_{\tilde{n}}^{(k)}, i, j) = (A_{2n}^{(2)}, 1, 1)$: the computations are a little more complicated than in case i), but substantially similar; we separate the cases $r = 2$ and $r > 2$ and, thanks to $(X2^+)$ and (HXL^+) , we get:

$$\begin{aligned}
 [\tilde{H}_{1,2}^+, X_{1,0}^+] & = (q - q^{-1}) k_1^{-1} ([2]_q k_1 X_{1,2}^+ + [[X_{1,2}^+, X_{1,0}^+]_{q^2}, X_{1,0}^-]) = \\
 & = (q - q^{-1}) k_1^{-1} ([2]_q k_1 X_{1,2}^+ + (q^4 - q^{-2}) [(X_{1,1}^+)^2, X_{1,0}^-]) = \\
 & = (q - q^{-1}) k_1^{-1} ([2]_q k_1 X_{1,2}^+ + (q^4 - q^{-2}) (X_{1,1}^+ [X_{1,1}^+, X_{1,0}^-] + [X_{1,1}^+, X_{1,0}^-] X_{1,1}^+)) = \\
 & = (q - q^{-1}) [2]_q X_{1,2}^+ + (q^4 - q^{-2}) (q^{-2} X_{1,1}^+ \tilde{H}_{1,1}^+ + \tilde{H}_{1,1}^+ X_{1,1}^+) = \\
 & = (q^2 - q^{-2}) X_{1,2}^+ + (q^4 - q^{-2}) \tilde{H}_{1,1}^+ X_{1,1}^+ + (q^2 - q^{-4}) X_{1,1}^+ \tilde{H}_{1,1}^+,
 \end{aligned}$$

hence, for all $s \in \mathbb{Z}$,

$$[\tilde{H}_{1,2}^+, X_{1,s}^+] = (q^2 - q^{-2}) X_{1,s+2}^+ + (q^4 - q^{-2}) [\tilde{H}_{1,1}^+, X_{1,s+1}^+]_{-q^{-2}};$$

$r > 2$:

$$[\tilde{H}_{1,r}^+, X_{1,0}^+] = (q - q^{-1}) k_1^{-1} ([2]_q k_1 X_{1,r}^+ + [[X_{1,r}^+, X_{1,0}^+]_{q^2}, X_{1,0}^-]) =$$

$$\begin{aligned}
&= (q - q^{-1})k_1^{-1}([2]_q k_1 X_{1,r}^+ + [(q^4 - q^{-2})[X_{1,r-1}^+, X_{1,1}^+]_{-1} - [X_{1,2}^+, X_{1,r-2}^+]_{q^2}, X_{1,0}^-]) = \\
&\quad = (q - q^{-1})k_1^{-1}([2]_q k_1 X_{1,r}^+ + \\
&\quad + (q^4 - q^{-2})[X_{1,r-1}^+, [X_{1,1}^+, X_{1,0}^-]]_{-1} + (q^4 - q^{-2})[X_{1,1}^+, [X_{1,r-1}^+, X_{1,0}^-]]_{-1} + \\
&\quad - [X_{1,2}^+, [X_{1,r-2}^+, X_{1,0}^-]]_{q^2} + q^2[X_{1,r-2}^+, [X_{1,2}^+, X_{1,0}^-]]_{q^{-2}}) = \\
&\quad = (q^2 - q^{-2})X_{1,r}^+ + \\
&\quad + (q^4 - q^{-2})[\tilde{H}_{1,1}^+, X_{1,r-1}^+]_{-q^{-2}} + (q^4 - q^{-2})[\tilde{H}_{1,r-1}^+, X_{1,1}^+]_{-q^{-2}} + \\
&\quad + q^2[\tilde{H}_{1,r-2}^+, X_{1,2}^+]_{q^{-4}} - [\tilde{H}_{1,2}^+, X_{1,r-2}^+] = \\
&\quad = (q^4 - q^{-2})\tilde{H}_{1,r-1}^+ X_{1,1}^+ + (q^2 - q^{-4})X_{1,1}^+ \tilde{H}_{1,r-1}^+ + \\
&\quad + q^2 \tilde{H}_{1,r-2}^+ X_{1,2}^+ - q^{-2} X_{1,2}^+ \tilde{H}_{1,r-2}^+;
\end{aligned}$$

this implies, using again (HXL^+) , that

$$\begin{aligned}
&({}_l \tilde{H}_1^+(u) - {}_r \tilde{H}_1^+(u))(X_{1,0}^+) = \\
&= ((q^4 - q^{-2}){}_l \tilde{H}_1^+(u) \tilde{t}_1 u + (q^2 - q^{-4}){}_r \tilde{H}_1^+(u) \tilde{t}_1 u + \\
&\quad + q^2 {}_l \tilde{H}_1^+(u) (\tilde{t}_1 u)^2 - q^{-2} {}_r \tilde{H}_1^+(u) (\tilde{t}_1 u)^2)(X_{1,0}^+),
\end{aligned}$$

or equivalently

$$\begin{aligned}
&{}_l \tilde{H}_1^+(u) (1 - q^4 \tilde{t}_1 u) (1 + q^{-2} \tilde{t}_1 u) (X_{1,0}^+) = \\
&= {}_r \tilde{H}_1^+(u) (1 - q^{-4} \tilde{t}_1 u) (1 + q^2 \tilde{t}_1 u) (X_{1,0}^+);
\end{aligned}$$

from this we get

$$\begin{aligned}
&(q - q^{-1})({}_l H_1(u) - {}_r H_1(u))(X_{1,0}^+) = \\
&= (\log(1 - q^{-4}(\tilde{t}_1 u)) + \log(1 + q^2(\tilde{t}_1 u)) - \log(1 - q^4(\tilde{t}_1 u)) - \log(1 + q^{-2}(\tilde{t}_1 u)))(X_{1,0}^+),
\end{aligned}$$

that is

$$[H_{1,r}, X_{1,0}^+] = \frac{-q^{-4r} + (-1)^{r-1} q^{2r} + q^{4r} - (-1)^{r-1} q^{-2r}}{r(q - q^{-1})} X_{1,r}^+ = b_{11r} X_{1,r}^+.$$

Proposition 16.

In $\tilde{\mathcal{U}}_q^{Dr}$ we have $\mathcal{I}(HH) \subseteq \mathcal{I}(HX)$.

Proof: Thanks to remark 3, to the fact that

$$[H_{i,r}, H_{j,s}] = -[H_{j,s}, H_{i,r}] = \tilde{\Omega}[H_{j,-s}, H_{i,-r}]$$

and to the definition of b_{ijr} , it is enough to prove that in $\tilde{\mathcal{U}}_q^{Dr}/\mathcal{I}(HX)$

$$[H_{i,r}, \tilde{H}_{j,s}^+] = \delta_{r+s,0} b_{ijr} (C^r - C^{-r}) \text{ if } |r| \geq s > 0.$$

This is an easy computation:

$$\begin{aligned}
[H_{i,r}, \tilde{H}_{j,s}^+] &= (q_j - q_j^{-1})k_j^{-1}[H_{i,r}, [X_{j,s}^+, X_{j,0}^-]] = \\
&= (q_j - q_j^{-1})k_j^{-1}([H_{i,r}, X_{j,s}^+], X_{j,0}^-] - [H_{i,r}, X_{j,0}^-], X_{j,s}^+]) =
\end{aligned}$$

$$\begin{aligned}
 &= (q_j - q_j^{-1})b_{ijr}k_j^{-1}(C^{\frac{r-|r|}{2}}[X_{j,r+s}^+, X_{j,0}^-] - C^{\frac{r+|r|}{2}}[X_{j,s}^+, X_{j,r}^-]) = \\
 &= b_{ijr}k_j^{-1}(C^{\frac{r-|r|}{2}}k_j\tilde{H}_{j,r+s}^+ - C^{\frac{-r-|r|}{2}-s}k_j^{-1}\tilde{H}_{j,r+s}^- + \\
 &\quad - C^{\frac{-r+|r|}{2}}k_j\tilde{H}_{j,r+s}^+ + C^{\frac{r+|r|}{2}-s}k_j^{-1}\tilde{H}_{j,r+s}^-) = \\
 &= b_{ijr}k_j^{-1}((C^{\frac{r-|r|}{2}} - C^{\frac{-r+|r|}{2}})k_j\tilde{H}_{j,r+s}^+ + (C^{\frac{r+|r|}{2}} - C^{\frac{-r-|r|}{2}})C^{-s}k_j^{-1}\tilde{H}_{j,r+s}^-) = \\
 &= b_{ijr}k_j^{-1}(C^{\frac{r-|r|}{2}} - C^{\frac{-r+|r|}{2}})k_j\tilde{H}_{j,r+s}^+ = \\
 &= \delta_{r+s,0}b_{ijr}(C^r - C^{-r}).
 \end{aligned}$$

Corollary 17.

- i) Relations (ZH) and (KH) are redundant.
- ii) In $\tilde{\mathcal{U}}_q^{Dr}$ relations (HX^\pm) depend on (XD^\pm) , $(X1^\pm)$ and $(X2^\pm)$ and relations (HH) depend on (XD) , $(X1)$ and $(X2)$.
- iii) $\mathcal{U}_q^{Dr}(X_{\tilde{n}}^{(k)})$ is the quotient of $\tilde{\mathcal{U}}_q^{Dr}(X_{\tilde{n}}^{(k)})$ by the ideal generated by the relations (XD^\pm) -($S3^\pm$).

Remark 18.

It is worth remarking that corollary 17,ii) allows us to reduce the relations (HX^\pm) and (HH) to relations involving just the $X_{i,r}^\pm$'s, without using the $H_{i,r}$'s whose connection with the $\tilde{H}_{i,r}^\pm$'s (these last can be expressed in terms of commutators between the $X_{i,r}^+$'s and the $X_{i,r}^-$'s, see remark §6.4) is complicated to handle. Indeed relations (HXL^\pm) can be translated as follows:

- i) If $\tilde{d}_i \leq |r| < \tilde{d}_{ij}$ then $[H_{i,r}, X_{j,s}^\pm] = 0$, that is $X_{j,s}^\pm$ commutes with the subalgebra generated by $\{H_{i,r} | \tilde{d}_i \leq |r| < \tilde{d}_{ij}\}$, which is $\{\tilde{H}_{i,r}^\pm | \tilde{d}_i \leq |r| < \tilde{d}_{ij}\}$ (see remark §6.3); hence these relations can be rewritten as

$$[\tilde{H}_{i,r}^\pm, X_{j,s}^\pm] = 0 \quad \text{and} \quad [\tilde{H}_{i,r}^\pm, X_{j,s}^\mp] = 0 \quad \text{if } |r| < \tilde{d}_{ij};$$

- ii) If $|r| = \tilde{d}_{ij}$ and $\pm r > 0$ then $\tilde{H}_{i,r}^\pm \mp (q_i - q_i^{-1})H_{i,r}$ commutes with $X_{j,s}^\pm$, by i) and remark §6.3, hence in the relations $[H_{i,r}, X_{j,s}^\pm] = \pm b_{ij\tilde{d}_{ij}}C^{\frac{r \mp |r|}{2}}X_{j,r+s}^\pm$ we can replace $H_{i,r}$ with $\pm \frac{\tilde{H}_{i,r}^\pm}{(q_i - q_i^{-1})}$.

Then relations (HXL^\pm) are equivalent to

$$[[X_{i,\pm r}^\pm, X_{i,0}^\mp], X_{j,s}^\pm]_{q_i^{a_{ij}}} = b_{ijr}k_i^{\pm 1}X_{j,\pm r}^\pm$$

and

$$[[X_{i,\pm r}^\pm, X_{i,0}^\mp], X_{j,s}^\mp]_{q_i^{-a_{ij}}} = -b_{ijr}C^{\pm r}k_i^{\pm 1}X_{j,\pm r}^\mp$$

with $0 < r \leq \tilde{d}_{ij}$.

Remark also that among the relations defining \mathcal{U}_q^{Dr} there are no other relations involving the $H_{i,r}$'s.

Remark 19.

Remark that $\forall i \in I_0$ $\{C^{\pm \tilde{d}_i}, k_i^{\pm 1}, X_{i,r}^{\pm}, H_{i,s} | \tilde{d}_i | r, s; s \neq 0\}$ generates $Im(\tilde{\phi}_i) \subseteq \tilde{\mathcal{U}}_q^{Dr}$ over $\mathbb{C}(q_i)$. Then the following sets generate $Im(\tilde{\phi}_i)$ (hence $Im(\phi_i) \subseteq \mathcal{U}_q^{Dr}$) over $\mathbb{C}(q_i)$:

- i) $\{C^{\pm \tilde{d}_i}, k_i^{\pm 1}, X_{i,r}^{\pm} | \tilde{d}_i | r\}$;
- ii) $\{C^{\pm \tilde{d}_i}, k_i^{\pm 1}, X_{i,0}^{\pm}, H_{i,\pm \tilde{d}_i}\}$;
- iii) $\{C^{\pm \tilde{d}_i}, k_i^{\pm 1}, X_{i,0}^{\pm}, X_{i,\mp \tilde{d}_i}^{\pm}\}$.

Moreover

- iv) $\{C^{\pm 1}, k_i^{\pm 1}, X_{i,0}^{\pm}, X_{i_0, \mp 1}^{\pm} | i \in I_0\}$ (where i_0 is any fixed element of I_0 with the property $\tilde{d}_{i_0} = 1$) generates $\tilde{\mathcal{U}}_q^{Dr}$ (hence \mathcal{U}_q^{Dr}) over $\mathbb{C}(q)$.

Proof:

- i) See remarks §6.3 and §6.4;
- ii) follows from i) by induction on $|r|$, using that

$$\forall r \in \mathbb{Z} \quad [H_{i,\pm \tilde{d}_i}, X_{i,r}^+] = b_{i\tilde{d}_i} C^{\frac{\pm 1 - 1}{2} \tilde{d}_i} X_{i,r \pm \tilde{d}_i}^+$$

and applying $\tilde{\Omega}$ (the set $\{C^{\pm \tilde{d}_i}, k_i^{\pm 1}, X_{i,0}^{\pm}, H_{i,\pm \tilde{d}_i}\}$ is $\tilde{\Omega}$ -stable);

iii) is an immediate consequence of ii) and of the fact that $[X_{i,\tilde{d}_i}^+, X_{i,0}^-] = k_i H_{i,\tilde{d}_i}$, again applying $\tilde{\Omega}$;

iv) $\forall i \in I_0$ there exists a sequence of different indices $i_0, i_1, \dots, i_l = i$ in I_0 such that $a_{i_{h-1}i_h} < 0$ and $\tilde{d}_{i_{h-1}} | \tilde{d}_{i_h} \quad \forall h = 1, \dots, l$.

We prove by induction on h that $Im(\phi_{i_h})$ is contained in the $\mathbb{C}(q)$ -subalgebra of $\tilde{\mathcal{U}}_q^{Dr}$ generated by $\{C^{\pm 1}, k_i^{\pm 1}, X_{i,0}^{\pm}, X_{i_0, \mp 1}^{\pm} | i \in I_0\}$, the claim for $h = 0$ being iii).

For $h > 0$ it is again enough to use iii), remarking that

$$[H_{i_{h-1}, -\tilde{d}_{i_h}}, X_{i_h,0}^+] = b_{i_{h-1}i_h \tilde{d}_{i_h}} C^{-\tilde{d}_{i_h}} X_{i_h, -\tilde{d}_{i_h}}^+ \neq 0$$

and applying $\tilde{\Omega}$.

§8. \mathcal{U}_q : (anti)AUTOMORPHISMS and RELATIONS.

The main point of this section is to describe in some details how the (anti)automorphisms $\tilde{\Omega}$, $\tilde{\Theta}$ and \tilde{t}_i ($i \in I_0$) act on the generators of the ideal of $\tilde{\mathcal{U}}_q$ defining \mathcal{U}_q . As a corollary $\tilde{\Omega}$, $\tilde{\Theta}$ and \tilde{t}_i ($i \in I_0$) induce analogous Ω , Θ and t_i ($i \in I_0$) on \mathcal{U}_q . But the important consequence of this analysis (together with the study of the commutation with the elements $H_{i,r}$'s) is the reduction of the huge amount of relations (XD^{\pm}) -($S3^{\pm}$) to relations involving only the positive $X_{i,r}^+$ (which is obvious and well known) and, which is new, to the analogous relations with “constant parameters” (see section §9). Lemmas §9.12 and

§9.14 are the fundamental tool of this paper, which makes possible and easy the computations of the following sections, leading to theorems §10.8 and §11.18.

Notation 1.

Let $l \in \mathbb{N}$; then:

- i) $\mathbf{1} = \mathbf{1}_l = (1, \dots, 1) \in \mathbb{Z}^l$;
- ii) $\{e_1, \dots, e_l\}$ is the canonical basis of \mathbb{Z}^l
- iii) for all $r = (r_1, \dots, r_l) \in \mathbb{Z}^l$ $\bar{r} \in \mathbb{Z}^l$ denotes $\bar{r} = (r_l, \dots, r_1)$.

Definition 2.

$\Omega : \mathcal{U}_q^{Dr} \rightarrow \mathcal{U}_q^{Dr}$ is the \mathbb{C} -anti-linear anti-homomorphism induced by $\tilde{\Omega}$ (and by $\bar{\Omega}$, see definition §6.15 and remark §6.16,vii), that is the \mathbb{C} -anti-linear anti-homomorphism defined on the generators by

$$q \mapsto q^{-1}, \quad C^{\pm 1} \mapsto C^{\mp 1}, \quad k_i^{\pm 1} \mapsto k_i^{\mp 1}, \quad X_{i,r}^{\pm} \mapsto X_{i,-r}^{\mp}, \quad H_{i,r} \mapsto H_{i,-r}.$$

Remark 3.

Ω is a well-defined involution of \mathcal{U}_q^{Dr} . Indeed

$$\begin{aligned} \tilde{\Omega}(M_{(2)}^{\pm}((i, r), (j, s))) &= -q_i^{-a_{ij}} M_{(2)}^{\mp}((i, -r), (j, -s)), \\ \tilde{\Omega}(M_i^{\pm}(r)) &= -q_i^{-2} M_i^{\mp}(-r), \\ \tilde{\Omega}(M_{(2,2)}^{\pm}(r)) &= -q^{-2} M_{(2,2)}^{\mp}(-r), \\ \tilde{\Omega}(M_{(3)}^{\varepsilon, \pm}(r)) &= q^{-6\varepsilon} M_{(3)}^{\varepsilon, \mp}(-r), \\ \tilde{\Omega}(M_{i,j;l;a}^{\pm}(r; s)) &= (-1)^l q_i^{l(a_{ij} + a(l-1))} M_{i,j;l;a}^{\mp}(-r; -s), \\ \tilde{\Omega}(X_{i,j;l;a}^{\pm}(r; s)) &= (-1)^l X_{i,j;l;a}^{\mp}(-\bar{r}; -s), \\ \tilde{\Omega}(X_{[k]}^{\pm}(r; s)) &= X_{[k]}^{\mp}(-\bar{r}; -s), \\ \tilde{\Omega}(M_{[2]}^{\pm}(r; s)) &= q^{-2} M_{[2]}^{\mp}(-r; -s), \\ \tilde{\Omega}(M_{[3]}^{\pm}(r; s)) &= q^{-4} M_{[3]}^{\mp}(-r; -s). \end{aligned}$$

Definition 4.

$\Theta : \mathcal{U}_q^{Dr} \rightarrow \mathcal{U}_q^{Dr}$ is the \mathbb{C} -anti-linear homomorphism induced by $\tilde{\Theta}$ (and by $\bar{\Theta}$, see definition §6.15 and corollary §7.13,ii), that is the \mathbb{C} -anti-linear homomorphism defined on the generators by

$$\begin{aligned} q \mapsto q^{-1}, \quad C^{\pm 1} \mapsto C^{\pm 1}, \quad k_i^{\pm 1} \mapsto k_i^{\mp 1}, \\ X_{i,r}^{+} \mapsto -X_{i,-r}^{+} k_i C^{-r}, \quad X_{i,r}^{-} \mapsto -k_i^{-1} C^{-r} X_{i,-r}^{-}, \quad H_{i,r} \mapsto H_{i,-r}. \end{aligned}$$

Remark 5.

Θ is a well-defined involution of \mathcal{U}_q^{Dr} . Indeed

$$\begin{aligned}\Theta_{\mathcal{F}}^{\pm}(M_{(2)}^{\pm}((i, r), (j, s))) &= -q_i^{-a_{ij}} M_{(2)}^{\pm}((i, -r \mp \tilde{d}_{ij}), (j, -s \mp \tilde{d}_{ij})), \\ \Theta_{\mathcal{F}}^{\pm}(M_i^{\pm}(r)) &= -q_i^{-2} M_i^{\pm}(-\bar{r} \mp \tilde{d}_{ij} \mathbf{1}), \\ \Theta_{\mathcal{F}}^{\pm}(M_{(2,2)}^{\pm}(r)) &= -q^{-2} M_{(2,2)}^{\pm}(-\bar{r} \mp 2\mathbf{1}), \\ \Theta_{\mathcal{F}}^{\pm}(M_{(3)}^{\varepsilon, \pm}(r)) &= M_{(3)}^{-\varepsilon, \pm}(-r), \\ \Theta_{\mathcal{F}}^{\pm}(X_{i,j;l;a}^{\pm}(r; s)) &= X_{i,j;l;a}^{\pm}(-r; -s), \\ \Theta_{\mathcal{F}}^{\pm}(X_{[k]}^{\pm}(r; s)) &= X_{[k]}^{\pm}(-r \mp (k-1)\mathbf{1}; -s).\end{aligned}$$

Definition 6.

For all $i \in I_0$ $t_i : \mathcal{U}_q^{Dr} \rightarrow \mathcal{U}_q^{Dr}$ is the $\mathbb{C}(q)$ -homomorphism induced by \tilde{t}_i (and by \bar{t}_i , see definition §6.15 and corollary §7.10,ii), that is the $\mathbb{C}(q)$ -homomorphism defined on the generators by

$$C^{\pm 1} \mapsto C^{\pm 1}, \quad k_j^{\pm 1} \mapsto (k_j C^{-\delta_{ij} \tilde{d}_i})^{\pm 1}, \quad X_{j,r}^{\pm} \mapsto X_{j,r \mp \delta_{ij} \tilde{d}_i}^{\pm}, \quad H_{j,r} \mapsto H_{j,r}.$$

Remark 7.

It is immediate to check that the t_i 's are well-defined automorphisms of \mathcal{U}_q^{Dr} . Indeed

$$\begin{aligned}\tilde{t}_i(M_{(2)}^{\pm}((j, r), (h, s))) &= M_{(2)}^{\pm}((j, r \mp \delta_{ij} \tilde{d}_i), (h, s \mp \delta_{ih} \tilde{d}_i)), \\ \tilde{t}_i(M_j^{\pm}(r)) &= M_j^{\pm}(r \mp \delta_{ij} \tilde{d}_i \mathbf{1}), \\ \tilde{t}_i(M_{(2,2)}^{\pm}(r)) &= M_{(2,2)}^{\pm}(r \mp \delta_{i1} \mathbf{1}), \\ \tilde{t}_i(M_{(3)}^{\varepsilon, \pm}(r)) &= M_{(3)}^{\varepsilon, \pm}(r \mp \delta_{i1} \mathbf{1}), \\ \tilde{t}_i(M_{j,h;l;a}^{\pm}(r; s)) &= M_{j,h;l;a}^{\pm}(r \mp \delta_{ij} \tilde{d}_i \mathbf{1}; s \mp \delta_{ih} \tilde{d}_i), \\ \tilde{t}_i(X_{j,h;l;a}^{\pm}(r; s)) &= X_{j,h;l;a}^{\pm}(r \mp \delta_{ij} \tilde{d}_i \mathbf{1}; s \mp \delta_{ih} \tilde{d}_i), \\ \tilde{t}_i(X_{[k]}^{\pm}(r; s)) &= X_{[k]}^{\pm}(r \mp \delta_{ij} \mathbf{1}; s \mp \delta_{ih} \tilde{d}), \\ \tilde{t}_i(M_{[k]}^{\pm}(r; s)) &= M_{[k]}^{\pm}(r \mp \delta_{ij} \mathbf{1}; s \mp \delta_{ih} k);\end{aligned}$$

in the last two identities $j, h \in I_0$ are such that $a_{jh} = -k$.

Remark 8.

Of course (see remark §6.16, v) and vi))

$$\Theta \Omega = \Omega \Theta, \quad t_i \Omega = \Omega t_i, \quad t_i \Theta = \Theta t_i^{-1} \quad \text{and} \quad t_i t_j = t_j t_i \quad \forall i, j \in I_0$$

as maps of $\mathcal{U}_q^{Dr}(X_{\tilde{n}}^{(k)})$ into itself, and, $\forall i \in I_0$,

$$\Omega \phi_i = \phi_i \Omega, \quad \Theta \phi_i = \phi_i \Theta, \quad t_i \phi_i = \phi_i t_1, \quad t_j \phi_i = \phi_i \quad \forall j \in I_0 \setminus \{i\}$$

as maps from $\mathcal{U}_q^{Dr}(A_1^{(1)})$ to $\mathcal{U}_q^{Dr}(X_{\tilde{n}}^{(k)})$ if $(X_{\tilde{n}}^{(k)}, i) \neq (A_{2n}^{(2)}, 1)$, from $\mathcal{U}_q^{Dr}(A_2^{(2)})$ to $\mathcal{U}_q^{Dr}(X_{\tilde{n}}^{(k)})$ if $(X_{\tilde{n}}^{(k)}, i) = (A_{2n}^{(2)}, 1)$;
for all $\alpha \in Q$, $\beta \in Q_0$, $m \in \mathbb{Z}$, $i \in I_0$ we have

$$\Omega(\mathcal{U}_{q,\alpha}^{Dr}) = \mathcal{U}_{q,-\alpha}^{Dr}, \quad \Theta(\mathcal{U}_{q,\beta+m\delta}^{Dr}) = \mathcal{U}_{q,\beta-m\delta}^{Dr}, \quad t_i(\mathcal{U}_{q,\alpha}^{Dr}) = \mathcal{U}_{q,\lambda_i(\alpha)}^{Dr}.$$

§9. REDUCTION to RELATIONS with CONSTANT PARAMETER.

We shall now apply the structures introduced until now to the analysis of the relations defining \mathcal{U}_q^{Dr} .

Notation 1.

Let (R) be relations as in notation §5.1 and define the following ideals of \mathcal{U} :

$$\mathcal{I}_{cte}(R) = (S_\zeta(r\mathbf{1}_l, s) | \zeta \in \mathcal{Z}, r \in \mathbb{Z}, s \in \mathbb{Z}^{\tilde{l}}),$$

$$\mathcal{I}_0(R) = (S_\zeta(\underline{0}) | \zeta \in \mathcal{Z}),$$

where $\underline{0} \in \mathbb{Z}^{l+\tilde{l}}$; more precisely, if $\#\mathcal{Z} = 1$ and $\mathcal{Z} = \{\zeta\}$, given $r \in \mathbb{Z}$ and $s \in \mathbb{Z}^{\tilde{l}}$, let

$$\mathcal{I}_{(r,s)}(R) (= \mathcal{I}_{(r,s)}(R_\zeta)) = (S_\zeta(r\mathbf{1}_l, s)).$$

If $(^{(h)}R)$ ($h = 1, \dots, m$) are as in notation §5.1,iii), define

$$\mathcal{I}_{cte}({}^{(1)}R, \dots, {}^{(m)}R) = (\mathcal{I}_{cte}({}^{(1)}R), \dots, \mathcal{I}_{cte}({}^{(m)}R))$$

and

$$\mathcal{I}_0({}^{(1)}R, \dots, {}^{(m)}R) = (\mathcal{I}_0({}^{(1)}R), \dots, \mathcal{I}_0({}^{(m)}R)).$$

Finally, if moreover each $(^{(h)}R) = (^{(h)}R^+) \cup (^{(h)}R^-)$ where $(^{(h)}R^\pm)$ is as in notation §5.1,iv), we shall also use the notation

$$\mathcal{I}_*^\pm({}^{(1)}R, \dots, {}^{(m)}R) = (\mathcal{I}_*({}^{(1)}R^\pm), \dots, \mathcal{I}_*({}^{(m)}R^\pm))$$

where $*$ $\in \{\emptyset, cte, 0\}$.

Remark 2.

With the notations fixed in notations §5.1 and 1 we have:

- i) $\mathcal{I}_0(R) \subseteq \mathcal{I}_{cte}(R) \subseteq \mathcal{I}(R)$;
- ii) $\mathcal{I}_*(R) = (\mathcal{I}_*(R_\zeta) | \zeta \in \mathcal{Z}) \forall * \in \{\emptyset, cte, 0\}$;
- iii) for all $\zeta \in \mathcal{Z}$ $\mathcal{I}_{cte}(R_\zeta) = (\mathcal{I}_{(r,s)}(R_\zeta) | (r, s) \in \mathbb{Z}^{1+\tilde{l}})$;
- iv) $l = 1 \Rightarrow \mathcal{I}_{cte}(R) = \mathcal{I}(R)$.

Remark 3.

Let (R^\pm) be relations in $\tilde{\mathcal{U}}_q^{Dr}$ as in notation §5.1,iv) and suppose that for all $\zeta \in \mathcal{Z}$ and for all $r \in \mathbb{Z}^l$, $s \in \mathbb{Z}^{\tilde{l}}$ there exists an invertible element $u_{\zeta,r,s}$ of $\tilde{\mathcal{U}}_q^{Dr}$ such that $\tilde{\Omega}(S_\zeta^+(r,s)) = u_{\zeta,r,s} S_\zeta^-(-r,-s)$ (notice that if (R^\pm) has this property then (R_ζ^\pm) has the same property). With the notations fixed in notations §5.1 and 1 we have:

- i) $\tilde{\Omega}(\mathcal{I}_{r,s}^+(R_\zeta)) = \mathcal{I}_{-r,-s}^-(R_\zeta) \forall \zeta \in \mathcal{Z}, (r,s) \in \mathbb{Z}^{1+\tilde{l}}$;
- ii) $\tilde{\Omega}(\mathcal{I}^+(R)) = \mathcal{I}^-(R)$, $\tilde{\Omega}(\mathcal{I}_{cte}^+(R)) = \mathcal{I}_{cte}^-(R)$ and $\tilde{\Omega}(\mathcal{I}_0^+(R)) = \mathcal{I}_0^-(R)$;
- iii) $\mathcal{I}(R)$, $\mathcal{I}_{cte}(R)$ and $\mathcal{I}_0(R)$ are the $\tilde{\Omega}$ -stable ideals generated respectively by $\mathcal{I}^+(R)$, $\mathcal{I}_{cte}^+(R)$ and $\mathcal{I}_0^+(R)$;

Remark 4.

Let (R) be relations in $\tilde{\mathcal{U}}_q^{Dr}$ as in notation §5.1 and suppose that for all $\zeta \in \mathcal{Z}$ there exist $i, j \in I_0$ such that for all $r \in \mathbb{Z}^l$, $s \in \mathbb{Z}^{\tilde{l}}$ we have:

$$\begin{aligned} \tilde{t}_i(S_\zeta(r,s)) &= S_\zeta(r - \mathbf{1}_l, s), \\ \tilde{t}_j(S_\zeta(r,s)) &= S_\zeta(r, s - \mathbf{1}_{\tilde{l}}) \\ \tilde{t}_h(S_\zeta(r,s)) &= S_\zeta(r,s) \quad \forall h \neq i, j. \end{aligned}$$

Then:

- i) $\mathcal{I}(R_\zeta)$ and $\mathcal{I}_{cte}(R_\zeta)$ are $\tilde{t}_i^{\pm 1}$ -stable for all $\zeta \in \mathcal{Z}$ and $i \in I_0$;
- ii) $\forall \zeta \in \mathcal{Z}$ and $\forall (r,s), (\tilde{r}, \tilde{s}) \in \mathbb{Z}^{1+\tilde{l}}$ there exists $\tilde{t} \in \langle \tilde{t}_i | i \in I_0 \rangle$ such that $\mathcal{I}_{(\tilde{r}, \tilde{s})}(R_\zeta) = \tilde{t}(\mathcal{I}_{(r,s)}(R_\zeta))$;
- iii) $\forall \zeta \in \mathcal{Z}$ and $\forall (r,s) \in \mathbb{Z}^{1+\tilde{l}}$ $\mathcal{I}_{cte}(R_\zeta)$ is the $\tilde{t}_i^{\pm 1}$ -stable (for all $i \in I_0$) ideal generated by $\mathcal{I}_{(r,s)}(R_\zeta)$;
- iv) $\mathcal{I}_{cte}(R)$ is the $\tilde{t}_i^{\pm 1}$ -stable ideal (for all $i \in I_0$) generated by $\mathcal{I}_0(R)$.

Corollary 5.

i) If (R) satisfies the conditions of remarks 3 and 4 then $\mathcal{I}_{cte}(R)$ is the $\tilde{\Omega}$ -stable and \tilde{t}_i -stable (for all $i \in I_0$) ideal generated by $\mathcal{I}_0^+(R)$. More precisely $\forall \zeta \in \mathcal{Z}$ $\mathcal{I}_{cte}(R_\zeta)$ is the $\tilde{\Omega}$ -stable and \tilde{t}_i -stable (for all $i \in I_0$) ideal generated by $\mathcal{I}_0^+(R_\zeta)$.

ii) Let $(^{(1)}R^\pm)$ and $(^{(2)}R^\pm)$ be as in remarks 3 and 4, and suppose that $\mathcal{I}_0^+(^{(1)}R) \subseteq \mathcal{I}_{cte}^+(^{(2)}R)$; then $\mathcal{I}_{cte}^+(^{(1)}R) \subseteq \mathcal{I}_{cte}^+(^{(2)}R)$.

More precisely $\mathcal{I}_{cte}^+(^{(1)}R) \subseteq \mathcal{I}_{cte}^+(^{(2)}R)$ if and only if for all $\zeta \in {}^{(1)}\mathcal{Z}$ there exists $(r,s) \in \mathbb{Z} \times \mathbb{Z}^{\tilde{l}_h}$ such that $\mathcal{I}_{(r,s)}^+(^{(1)}R_\zeta) \subseteq \mathcal{I}_{cte}^+(^{(2)}R)$, and if this is the case we have also $\mathcal{I}_{cte}^-(^{(1)}R) \subseteq \mathcal{I}_{cte}^-(^{(2)}R)$.

Remark 6.

With the notations fixed in notation §5.1 suppose that

$$\sigma.S_\zeta(r,s) = S_\zeta(r,s) \quad \forall \zeta \in \mathcal{Z}, r \in \mathbb{Z}^l, s \in \mathbb{Z}^{\tilde{l}}, \sigma \in \mathcal{S}_l,$$

where $\sigma.S_\zeta(r, s) = S_\zeta(\sigma.r, s)$, see notation §4.1,iv).

This condition is equivalent to the existence of elements $N_\zeta(r, s) \in \tilde{\mathcal{U}}_q^{Dr}$ such that

$$S_\zeta(r, s) = \sum_{\sigma \in \mathcal{S}_l} \sigma.N_\zeta(r, s).$$

Notice that in general the elements $N_\zeta(r, s)$ ($r \in \mathbb{Z}^l$, $s \in \mathbb{Z}^{\tilde{l}}$) are not uniquely determined by the $S_\zeta(r, s)$'s.

But $N_\zeta(r\mathbf{1}_l, s) = \frac{1}{\hbar} S_\zeta(r\mathbf{1}_l, s)$ for all $(r, s) \in \mathbb{Z} \times \mathbb{Z}^{\tilde{l}}$.

Remark 7.

In the hypotheses of remark 6 suppose that:

$$\text{i) } \mathcal{Z} \subseteq \begin{cases} I_0 & \text{if } \tilde{l} = 0 \\ \{(i, j) \in I_0 \times I_0 \mid i \neq j\} & \text{if } \tilde{l} = 1; \end{cases}$$

ii) if $\tilde{l} = 0$ and $i \in \mathcal{Z} \subseteq I_0$ there exists $c_{p,\sigma} \in \mathbb{C}(q)$ ($p \in \mathbb{Z}^l$, $\sigma \in \mathcal{S}_l$) such that for all $r \in \mathbb{Z}^l$

$$N_i(r) = \sum_{\substack{p \in \mathbb{Z}^l \\ \sigma \in \mathcal{S}_l}} c_{p,\sigma} X_{i, \tilde{d}_i(r_{\sigma(1)} + p_1)}^+ \cdot \dots \cdot X_{i, \tilde{d}_i(r_{\sigma(l)} + p_l)}^+;$$

iii) if $\tilde{l} = 1$ and $(i, j) \in \mathcal{Z} \subseteq I_0 \times I_0$ there exists $\tilde{c}_{p,\sigma,u} \in \mathbb{C}(q)$ ($p \in \mathbb{Z}^l$, $\sigma \in \mathcal{S}_l$, $u \in \{0, \dots, l\}$) such that for all $(r, s) \in \mathbb{Z}^l \times \mathbb{Z}^{\tilde{l}}$

$$\begin{aligned} & N_{(i,j)}(r, s) = \\ &= \sum_{\substack{p \in \mathbb{Z}^l, \sigma \in \mathcal{S}_l \\ u=0, \dots, l}} \tilde{c}_{p,\sigma,u} X_{i, \tilde{d}_i(r_{\sigma(1)} + p_1)}^+ \dots X_{i, \tilde{d}_i(r_{\sigma(u)} + p_u)}^+ X_{j, \tilde{d}_j s}^+ X_{i, \tilde{d}_i(r_{\sigma(u+1)} + p_{u+1})}^+ \dots X_{i, \tilde{d}_i(r_{\sigma(l)} + p_l)}^+. \end{aligned}$$

Then the conditions of remark 4 are satisfied.

Remark 8.

The relations $(XD^\pm)-(S3^\pm)$, as well as (Tk^\pm) and (S^\pm) , are of the form described in remark 7 and satisfy the hypotheses of remark 3, so that they all satisfy the conditions of remarks 3 and 4 and in particular the properties stated in colollary 5,i).

Remark 9.

If the relations (R) are of the form described in remarks 6 and 7 we have that $\forall h \in I_0, p \in \mathbb{Z}_+$:

$$[H_{h,p}, S_i(r)] = b_{hip} \sum_{u=1}^l S_i(r + \frac{p}{\tilde{d}_i} e_u) \quad \text{if } \tilde{l} = 0,$$

$$[H_{h,p}, S_{(i,j)}(r, s)] = b_{hjp} S_{(i,j)}(r, s + \frac{p}{d_j}) + b_{hip} \sum_{u=1}^l S_{(i,j)}(r + \frac{p}{d_i} e_u, s) \text{ if } \tilde{l} = 1,$$

where $S_\zeta(r, s) = 0$ if $(r, s) \notin \mathbb{Z}^{l+\tilde{l}}$.

Our next goal is studying the ideals $\mathcal{I}^\pm(R)$ and $\mathcal{I}(R)$ (see notations §5.1 and 1), providing a set of generators smaller and simpler than all of $\{S_\zeta^\pm(r, s) | \zeta \in \mathcal{Z}, r \in \mathbb{Z}^l, s \in \mathbb{Z}^{\tilde{l}}\}$. More precisely we shall show that under suitable hypotheses (fulfilled by the relations defining \mathcal{U}_q^{Dr} over $\tilde{\mathcal{U}}_q^{Dr}$) we have $\mathcal{I}^\pm(R) = \mathcal{I}_{cte}^\pm(R)$.

Remark 10.

The relations (XD^+) satisfy the conditions of remark 2,iv), hence in particular $\mathcal{I}(XD^+) = \mathcal{I}_{cte}(XD^+)$ and $\mathcal{I}(XD) = \mathcal{I}_{cte}(XD)$ (see remarks 3 and 8).

We shall generalize in two steps this result for (XD^\pm) to relations (R) satisfying the properties described in remarks 3 and 7: the cases $l = 2, \tilde{l} = 0$ (in particular $(X1^\pm)$ and $(X2^\pm)$) shall follow from lemma 12, while the general case will be an application of lemma 14.

Remark that if we considered $\tilde{\mathcal{U}}_q^{Dr}/\mathcal{I}(HX)$ instead of $\tilde{\mathcal{U}}_q^{Dr}$ we would not need to deal with the two cases separately, but the result would follow in both cases from lemma 14.

For the next remark recall notation §8.1.

Remark 11.

Consider an algebra \mathcal{U} over a field of characteristic 0, an automorphism t of \mathcal{U} , and elements $z, N(r) \in \mathcal{U}$ ($r \in \mathbb{Z}^2$), such that:

- i) $t(N(r)) = N(r + \mathbf{1}) \forall r \in \mathbb{Z}^2$;
- ii) $[z, N(r)] = N(r + e_1) + N(r + e_2) = N(r_1 + 1, r_2) + N(r_1, r_2 + 1) \forall r = (r_1, r_2) \in \mathbb{Z}^2$.

If we put $S(r) = \sum_{\sigma \in \mathcal{S}_2} N(\sigma(r))$ then of course:

- a) $S(r) = S(\bar{r}) \forall r \in \mathbb{Z}^2$;
- b) $S(r)$ satisfies i) and ii);
- c) $S(0) = 2N(0)$.

Lemma 12.

Let $\mathcal{U}, t, z, N(r), S(r)$ be as in remark 11.

If $N(0) = 0$ then $S(r) = 0 \forall r \in \mathbb{Z}^l$.

Proof: First of all remark 11 implies that it is enough to prove that $S(0, r) = 0 \forall r \in \mathbb{N}$: indeed a) of remark 11 implies that one can suppose $r_1 \leq r_2$; moreover applying t^{-r_1} one reduces to the case $r_1 = 0$.

Let us proceed by induction on r : if $r = 0$ the claim is true by hypothesis; let $r > 0$; then by the inductive hypothesis $S(0, r-1) = 0$ and $0 = [z, S(0, r-1)] = S(1, r-1) + S(0, r)$; if $r = 1$ $S(1, r-1) + S(0, r) = S(1, 0) + S(0, 1) = 2S(0, 1)$, so that $S(0, 1) = 0$; if $r > 1$ then $S(1, r-1) = t(S(0, r-2)) = 0$ by the inductive hypothesis, so that also $S(0, r) = 0$.

Remark 13.

Consider an algebra \mathcal{U} over a field of characteristic 0, an automorphism t of \mathcal{U} , elements $z_m, N_y(r) \in \mathcal{U}$ ($m \in \mathbb{Z}_+$, $y \in \mathcal{U}$, $r \in \mathbb{Z}^l$ with $l \in \mathbb{Z}_+$ fixed), such that:

- i) $t(N_y(r)) = N_y(r + \mathbf{1}) \forall y \in \mathcal{U}, r \in \mathbb{Z}^l$;
 - ii) $[z_m, N_y(r)] = N_{[z_m, y]}(r) + \sum_{u=1}^l N_y(r + me_u)$.
- If we put $S_y(r) = \sum_{\sigma \in \mathcal{S}_l} N_y(\sigma(r))$ then of course:
- a) $S_y(r) = S_y(\sigma(r)) \forall \sigma \in \mathcal{S}_l$;
 - b) $S_y(r)$ satisfies ii) and iii);
 - c) $S_y(0) = l!N_y(0)$.

Lemma 14.

Let \mathcal{U} , t , z_m , $N_y(r)$, $S_y(r)$ be as in remark 13 and let $Y \subseteq \mathcal{U}$ be a subset such that $[z_m, Y] \subseteq Y \forall m \in \mathbb{Z}_+$.

If $N_y(0) = 0 \forall y \in Y$ then $S_y(r) = 0 \forall y \in Y \forall r \in \mathbb{Z}^l$.

Proof: First of all remark 13 implies that it is enough to prove that $S_y(r) = 0 \forall y \in Y \forall r = (r_1, \dots, r_l) \in \mathbb{Z}^l$ such that $0 = r_1 \leq \dots \leq r_l$: indeed a) of remark 13 implies that one can suppose $r_1 \leq \dots \leq r_l$; moreover applying t^{-r_1} one reduces to the case $r_1 = 0$.

Let $v = \max\{u = 1, \dots, l | r_u = 0\}$ and proceed by induction on v : if $v = l$ then $r = 0$ and the claim is true by hypothesis; let $v < l$ and choose $m = r_{v+1}$; then

$$\max\{u = 1, \dots, l | (r - me_{v+1})_u = 0\} = v + 1$$

and

$$\max\{u = 1, \dots, l | (r - me_{v+1} + me_u)_u = 0\} = v + 1 \quad \forall u > v + 1,$$

hence by the inductive hypothesis $S_y(r - me_{v+1}) = 0$ and $S_y(r - me_{v+1} + me_u) = 0 \forall y \in Y \forall u > v + 1$; it follows that

$$\begin{aligned} 0 = [z_m, S_y(r - me_{v+1})] &= S_{[z_m, y]}(r - me_{v+1}) + \sum_{u=1}^l S_y(r - me_{v+1} + me_u) = \\ &= \sum_{u=1}^{v+1} S_y(r - me_{v+1} + me_u) = (v + 1)S_y(r). \end{aligned}$$

Proposition 15.

Consider the notations fixed in notation §5.1 and suppose that (R^+) satisfies the hypotheses of remark 7. Then:

- i) if $l = 2$ and $\tilde{l} = 0$ we have $\mathcal{I}^+(R) = \mathcal{I}_{cte}^+(R)$ in $\tilde{\mathcal{U}}_q^{Dr}(X_{\tilde{n}}^{(k)})$;
- ii) in any case $\mathcal{I}^+(R) = \mathcal{I}_{cte}^+(R)$ in $\tilde{\mathcal{U}}_q^{Dr}(X_{\tilde{n}}^{(k)})/\mathcal{I}(\mathcal{HX}^+)$;
- iii) if moreover the hypotheses of remark 3 are satisfied then we have also $\mathcal{I}^-(R) = \mathcal{I}_{cte}^-(R)$ (in $\tilde{\mathcal{U}}_q^{Dr}(X_{\tilde{n}}^{(k)})$ and in $\tilde{\mathcal{U}}_q^{Dr}(X_{\tilde{n}}^{(k)})/\mathcal{I}(HX^-)$ respectively).

Proof: Let $\zeta \in \mathcal{Z}$; thanks to remarks 4,iv) and §7.9, for all $i \in I_0$ \tilde{t}_i induces an automorphism t'_i of $\tilde{\mathcal{U}}_q^{Dr}(X_{\tilde{n}}^{(k)})/\mathcal{I}_{cte}^+(R_\zeta)$ and of $\tilde{\mathcal{U}}_q^{Dr}(X_{\tilde{n}}^{(k)})/\mathcal{I}_{cte}^+(HX, R_\zeta)$.

- i) Fix $i \in \mathcal{Z} \subseteq I_0$, and notice that the data

$$\mathcal{U} = \tilde{\mathcal{U}}_q^{Dr}/\mathcal{I}_{cte}^+(R_i), \quad t = t_i'^{-1}, \quad z = \frac{1}{b_{ii\tilde{d}_i}} H_{(i,\tilde{d}_i)}, \quad N(r) = N_i(r) \quad (r \in \mathbb{Z}^2)$$

satisfy the conditions of remark 11 and lemma 12.

Since we have that $\mathcal{I}_0^+(R_i) = 0$ in $\tilde{\mathcal{U}}_q^{Dr}/\mathcal{I}_{cte}^+(R_i)$, lemma 12 implies that $\mathcal{I}^+(R_i) = 0$ in $\tilde{\mathcal{U}}_q^{Dr}/\mathcal{I}_{cte}^+(R_i)$, or equivalently that in $\tilde{\mathcal{U}}_q^{Dr}$ $\mathcal{I}^+(R_i) \subseteq \mathcal{I}_{cte}^+(R_i)$, and this for all $i \in I_0 \subseteq \mathcal{Z}$, so that $\mathcal{I}^+(R) = \mathcal{I}_{cte}^+(R)$ thanks to remark 2,i) and ii).

- ii) Fix $\zeta = \begin{cases} i \in \mathcal{Z} \subseteq I_0 & \text{if } \tilde{l} = 0 \\ (i, j) \in \mathcal{Z} \subseteq I_0 \times I_0 & \text{if } \tilde{l} = 1 \end{cases}$, and notice that the data

$$\mathcal{U} = \tilde{\mathcal{U}}_q^{Dr}/\mathcal{I}_{cte}^+(HX, R_\zeta), \quad t = t_i'^{-1}, \quad z_m = \frac{1}{b_{ii\tilde{d}_m}} H_{(i,\tilde{d}_m)} \quad \forall m \in \mathbb{Z}_+,$$

$$Y = \begin{cases} \{0, 1\} & \text{if } \tilde{l} = 0 \\ \{aX_{j,\tilde{d}_j}^+ | s \in \mathbb{Z}, a \in \mathbb{C}(q)\} & \text{if } \tilde{l} = 1, \end{cases}$$

and for $r \in \mathbb{Z}^l$

$$N_y(r) = \begin{cases} yN_i^+(r) & \text{if } \tilde{l} = 0 \\ aN_{(i,j)}^+(r, s) & \text{if } \tilde{l} = 1, y = aX_{j,\tilde{d}_j}^+ \end{cases}$$

satisfy the conditions of remark 13 and lemma 14.

Since we have that $\mathcal{I}_0^+(R_i) = 0$ in $\tilde{\mathcal{U}}_q^{Dr}/\mathcal{I}_{cte}^+(HX, R_i)$ if $\tilde{l} = 0$ and $\mathcal{I}_{0,s}^+(R_{(i,j)}) = 0$ in $\tilde{\mathcal{U}}_q^{Dr}/\mathcal{I}_{cte}^+(HX, R_{(i,j)})$ for all $s \in \mathbb{Z}$ if $\tilde{l} = 1$, lemma 14 implies that $\mathcal{I}^+(R_\zeta) = 0$ in $\tilde{\mathcal{U}}_q^{Dr}/\mathcal{I}_{cte}^+(HX, R_\zeta)$, or equivalently that in $\tilde{\mathcal{U}}_q^{Dr}/\mathcal{I}_{cte}^+(HX)$ $\mathcal{I}^+(R_\zeta) \subseteq \mathcal{I}_{cte}^+(R_\zeta)$, so that $\mathcal{I}^+(R) = \mathcal{I}_{cte}^+(R)$ thanks to remark 2,i) and ii).

It follows that $\mathcal{I}^+(R) \subseteq (\mathcal{I}^+(HX), \mathcal{I}_{cte}^+(R))$.

- iii) follows from i) and ii) thanks to remarks 3 and §7.3.

Remark 16.

In proposition 15,i) the hypothesis $\tilde{l} = 0$ is not necessary: the claim would hold also in case $\tilde{l} = 1$. But this case is omitted here because it is not really needed in this paper and its proof, very similar, would just require a little more complicated, and repetitive, exposition (see the proof of proposition 15,ii)).

Corollary 17.

- i) $\mathcal{I}^\pm(X1) = \mathcal{I}_{cte}^\pm(X1)$ and $\mathcal{I}^\pm(X2) = \mathcal{I}_{cte}^\pm(X2)$.
- ii) If (R^\pm) is one of $(X3^{\varepsilon,\pm})$ -($S3^\pm$), (Tk^\pm) and (S^\pm) then

$$\mathcal{I}^\pm(R) \subseteq (\mathcal{I}^\pm(HX), \mathcal{I}_{cte}^\pm(R)).$$

- iii) If (R^\pm) is one of $(X3^{\varepsilon,\pm})$ -($S3^\pm$) and (S^\pm) then

$$\mathcal{I}^\pm(R, XD, X1, X2) = \mathcal{I}_{cte}^\pm(R, XD, X1, X2).$$

Proof: The claims follow from proposition 15, remarks 8 and 10 and proposition §7.15.

Remark 18.

Remark 16 would imply that furthermore $\mathcal{I}^\pm(R) = \mathcal{I}_{cte}^\pm(R)$ even in the case when (R^\pm) is one of (Sk^\pm) , (Tk^\pm) and, if $a_{ij} = -1$, also $(S(UL)_{(i,j)}^\pm)$.

Corollary 19.

Proposition 15 implies that

$$\mathcal{U}_q^{Dr} = \tilde{\mathcal{U}}_q^{Dr} / \mathcal{I}_{cte}(XD, X1, X2, X3^\varepsilon, SUL, S2, S3).$$

The final remark of this section is a straightforward consequence of remark §8.5.

Remark 20.

- i) If (R^\pm) is one of (XD^\pm) -($X2^\pm$), (SUL^\pm) -($S3^\pm$), (Tk^\pm) , (S^\pm) then $\mathcal{I}^+(R)$ and $\mathcal{I}_{cte}^+(R)$ are $\tilde{\Theta}$ -stable.
- ii) $\mathcal{I}^+(X3^1, X3^{-1})$ and $\mathcal{I}_{cte}^+(X3^1, X3^{-1})$ are the $\tilde{\Theta}$ -stable ideal generated respectively by $\mathcal{I}^+(X3^1)$ and $\mathcal{I}_{cte}^+(X3^1)$;
- iii) $\mathcal{I}_{cte}^+(X3^1, X3^{-1})$ is the $\tilde{\Theta}$ -stable and t_i -stable ideal (for all $i \in I_0$) generated by $\mathcal{I}_0^+(X3^1)$;
- iv) $\mathcal{I}(X3^1, X3^{-1}, X2) = \mathcal{I}_{cte}(X3^1, X3^{-1}, X2)$ is the $\tilde{\Omega}$ -stable, $\tilde{\Theta}$ -stable and \tilde{t}_i -stable ideal (for all $i \in I_0$) generated by $\mathcal{I}_0^+(X3^1, X2)$.

§10. MORE about REDUNDANT RELATIONS.

In this section we prove some dependences among the relations (XD^\pm) - $(S3^\pm)$ making systematic recourse to the properties of q -commutators (remark §4.17) and to corollary §9.5,ii).

Proposition 1.

With the notations of remark §9.1, $\mathcal{I}_{cte}^\pm(X2), \mathcal{I}_{cte}^\pm(X3^{-1}) \subseteq \mathcal{I}_{cte}^\pm(X3^1)$.

Proof: $\mathcal{I}_0^+(X2) \subseteq \mathcal{I}_{cte}^+(X3^1)$:

$$[[[X_{1,1}^+, X_{1,0}^+]_{q^2}, X_{1,0}^+]_{q^4}, X_{1,-1}^-] \in \mathcal{I}_{cte}^+(X3^1);$$

but

$$\begin{aligned} & [[X_{1,1}^+, X_{1,0}^+]_{q^2}, X_{1,0}^+]_{q^4}, X_{1,-1}^-] = \left[\left[\frac{Ck_1 - C^{-1}k_1^{-1}}{q - q^{-1}}, X_{1,0}^+ \right]_{q^2}, X_{1,0}^+ \right]_{q^4} + \\ & + [[X_{1,1}^+, k_1^{-1}H_{1,-1}]_{q^2}, X_{1,0}^+]_{q^4} + [[X_{1,1}^+, X_{1,0}^+]_{q^2}, k_1^{-1}H_{1,-1}]_{q^4} = \\ & = k_1^{-1}(q^2[2]_q C^{-1}[X_{1,0}^+, X_{1,0}^+]_{q^6} - q^2[[H_{1,-1}, X_{1,1}^+], X_{1,0}^+]_{q^6} - q^4[H_{1,-1}, [X_{1,1}^+, X_{1,0}^+]_{q^2}]) = \\ & = -q^4[3]_q! C^{-1}k_1^{-1}([X_{1,1}^+, X_{1,-1}^+]_{q^2} - (q^4 - q^{-2})(X_{1,0}^+)^2) = -q^4[3]_q! C^{-1}k_1^{-1}M_{(2,2)}^+(0); \end{aligned}$$

then $\mathcal{I}_0^+(X2) \subseteq \mathcal{I}_{cte}^+(X3^1)$.

$\mathcal{I}_0^+(X3^{-1}) \subseteq \mathcal{I}_{cte}^+(X3^1)$: notice that

$$\begin{aligned} & \mathcal{I}_{cte}^+(X3^1) \ni \frac{1}{b_{i\tilde{i}\tilde{d}_i}} \tilde{t}_1([H_{1,1}, M_{(3)}^{1,+}(0)]) = \\ & = [[X_{1,1}^+, X_{1,-1}^+]_{q^2}, X_{1,-1}^+]_{q^4} + [[X_{1,0}^+, X_{1,0}^+]_{q^2}, X_{1,-1}^+]_{q^4} + [[X_{1,0}^+, X_{1,-1}^+]_{q^2}, X_{1,0}^+]_{q^4} = \\ & = [[X_{1,1}^+, X_{1,-1}^+]_{q^2} - (q^4 - q^{-2})(X_{1,0}^+)^2, X_{1,-1}^+]_{q^4} + (q^4 - q^{-2} + 1 - q^2 - q^4)(X_{1,0}^+)^2 X_{1,-1}^+ + \\ & + (1 + q^6)X_{1,0}^+ X_{1,-1}^+ X_{1,0}^+ + (-q^8 + q^2 - q^4 + q^6 - q^2)X_{1,-1}^+ (X_{1,0}^+)^2, \end{aligned}$$

so that

$$\begin{aligned} & -(q^2 - 1 + q^{-2})((X_{1,0}^+)^2 X_{1,-1}^+ - (q^4 + q^2)X_{1,0}^+ X_{1,-1}^+ X_{1,0}^+ + q^6 X_{1,-1}^+ (X_{1,0}^+)^2) = \\ & = -q^6(q^2 - 1 + q^{-2})[[X_{1,-1}^+, X_{1,0}^+]_{q^{-2}}, X_{1,0}^+]_{q^{-4}} = -q^6(q^2 - 1 + q^{-2})M_{(3)}^{-1,+}(0) \end{aligned}$$

is an element of $\mathcal{I}_{cte}^+(X3^1)$. The claims follow again from corollary §9.5.

Corollary 2.

- i) $\mathcal{I}^\pm(X3^1) = \mathcal{I}_{cte}^\pm(X3^1)$ (see corollary §9.17,iii)).
- ii) $\mathcal{I}_{cte}^+(X3^1) = \mathcal{I}_{cte}^+(X3^{-1})$ is $\tilde{\Theta}$ -stable.
- iii) Moreover $(\mathcal{I}_{cte}^+(X3^1), \mathcal{I}_{cte}^-(X3^{-1})) = \mathcal{I}_{cte}(X3^1) = \mathcal{I}_{cte}(X3^{-1})$ is $\tilde{\Omega}$ -stable.

Proposition 3.

- i) $\mathcal{U}_q^{Dr}(A_1^{(1)}) = \tilde{\mathcal{U}}_q^{Dr}(A_1^{(1)})/\mathcal{I}_{cte}(X1)$;

$$\text{ii) } \mathcal{U}_q^{Dr}(A_2^{(2)}) = \tilde{\mathcal{U}}_q^{Dr}(A_2^{(2)})/\mathcal{I}_{cte}(X3^1) = \tilde{\mathcal{U}}_q^{Dr}(A_2^{(2)})/(\mathcal{I}_{cte}^+(X3^1), \mathcal{I}_{cte}^-(X3^{-1})).$$

Proposition 4.

$$\mathcal{I}_{cte}^\pm(XD) \subseteq \mathcal{I}_{cte}^\pm(SUL).$$

Proof: Let $i, j \in I_0$ be such that $a_{ij} < 0$; since $-1 \in \{a_{ij}, a_{ji}\}$, in the study of $[X_{i, \tilde{d}_{ij}}^+, X_{j, 0}^+]_{q_i^{a_{ij}}} + [X_{j, \tilde{d}_{ij}}^+, X_{i, 0}^+]_{q_i^{a_{ij}}}$ we can suppose that $a_{ij} = -1$, and in particular $\tilde{d}_j \leq \tilde{d}_i = \tilde{d}_{ij}$ and, if $X_{\tilde{n}}^{(k)} = A_{2n}^{(2)}$, $i \neq 1$. Then $[[X_{j, 0}^+, X_{i, 0}^+]_q, X_{i, 0}^+]_{q^{-1}}$ is an element of $\mathcal{I}_{cte}^+(SUL)$, and so is $[[X_{j, 0}^+, X_{i, 0}^+]_q, X_{i, 0}^+]_{q^{-1}}, X_{i, \tilde{d}_i}^-]$. But

$$\begin{aligned} & [[X_{j, 0}^+, X_{i, 0}^+]_q, X_{i, 0}^+]_{q^{-1}}, X_{i, \tilde{d}_i}^-] = \\ & = [[X_{j, 0}^+, X_{i, 0}^+]_q, C^{-\tilde{d}_i} k_i H_{i, \tilde{d}_i}]_{q^{-1}} + [[X_{j, 0}^+, C^{-\tilde{d}_i} k_i H_{i, \tilde{d}_i}]_q, X_{i, 0}^+]_{q^{-1}} = \\ & = C^{-\tilde{d}_i} k_i (q^{-1} [[X_{j, 0}^+, X_{i, 0}^+]_q, H_{i, \tilde{d}_i}] + q [[X_{j, 0}^+, H_{i, \tilde{d}_i}]_q, X_{i, 0}^+]_{q^{-3}}) = \\ & = C^{-\tilde{d}_i} k_i (-b_{ii\tilde{d}_i} q^{-1} [X_{j, 0}^+, X_{i, \tilde{d}_i}^+]_q - b_{ij\tilde{d}_i} q^{-1} [X_{j, \tilde{d}_i}^+, X_{i, 0}^+]_q - b_{ij\tilde{d}_i} q [X_{j, \tilde{d}_i}^+, X_{i, 0}^+]_{q^{-3}}) = \\ & = [2]_{q_i} C^{-\tilde{d}_i} k_i ([X_{i, \tilde{d}_i}^+, X_{j, 0}^+]_{q^{-1}} + [X_{j, \tilde{d}_i}^+, X_{i, 0}^+]_{q^{-1}}). \end{aligned}$$

Then $\mathcal{I}_0^+(XD) \subseteq \mathcal{I}_{cte}^+(SUL)$, and the claim follows using corollary §9.5.

Lemma 5.

For $i \in I_0$, $a \in \mathbb{N}$ define $Y_{i, a} \in \tilde{\mathcal{U}}_q^{Dr}(X_{\tilde{n}}^{(k)})$ as follows:

$$Y_{i, 0} = X_{i, \tilde{d}_i}^+; \quad Y_{i, a+1} = [Y_{i, a}, X_{i, 0}^+]_{q_i^{2(a+1)}}.$$

(Notice that $Y_{i, 1} = M_i^+(0)$).

Then:

- i) $[Y_{i, a}, X_{i, 0}^-] = (b_{ii\tilde{d}_i} - [a]_{q_i} [a+1]_{q_i}) k_i Y_{i, a-1} \quad \forall a > 0$;
- ii) $[X_{i, j; 1-a_{ij}, 1}^+(0; 0), X_{j, \tilde{d}_{ij}}^-] = C^{-\tilde{d}_{ij}} k_j b_{ji\tilde{d}_{ij}} Y_{i, -a_{ij}}$.

Proof: i) $[Y_{i, a}, X_{i, 0}^-] =$

$$\begin{aligned} & = [[\dots [X_{i, \tilde{d}_i}^+, X_{i, 0}^+]_{q_i^2}, \dots X_{i, 0}^+]_{q_i^{2u}}, \dots X_{i, 0}^+]_{q_i^{2a}}, X_{i, 0}^-] = \\ & = [\dots [k_i H_{i, \tilde{d}_i}, X_{i, 0}^+]_{q_i^2}, \dots X_{i, 0}^+]_{q_i^{2u}}, \dots X_{i, 0}^+]_{q_i^{2a}} + \\ & + \sum_{u=1}^a [\dots [\dots [X_{i, \tilde{d}_i}^+, X_{i, 0}^+]_{q_i^2}, \dots X_{i, 0}^+]_{q_i^{2(u-1)}}, \frac{k_i - k_i^{-1}}{q_i - q_i^{-1}}]_{q_i^{2u}}, X_{i, 0}^+]_{q_i^{2(u+1)}}, \dots X_{i, 0}^+]_{q_i^{2a}} = \\ & = k_i [\dots [\dots [H_{i, \tilde{d}_i}, X_{i, 0}^+]_{q_i^2}, \dots X_{i, 0}^+]_{q_i^{2(u-1)}}, \dots X_{i, 0}^+]_{q_i^{2(a-1)}} + \\ & + \sum_{u=1}^a \frac{q_i^{-2u} - q_i^{2u}}{q_i - q_i^{-1}} k_i [\dots [\dots [X_{i, \tilde{d}_i}^+, X_{i, 0}^+]_{q_i^2}, \dots X_{i, 0}^+]_{q_i^{2(u-1)}}, X_{i, 0}^+]_{q_i^{2u}}, \dots X_{i, 0}^+]_{q_i^{2(a-1)}} = \\ & = b_{ii\tilde{d}_i} k_i [\dots [\dots [X_{i, \tilde{d}_i}^+, X_{i, 0}^+]_{q_i^2}, \dots X_{i, 0}^+]_{q_i^{2(u-1)}}, \dots X_{i, 0}^+]_{q_i^{2(a-1)}} + \end{aligned}$$

$$\begin{aligned}
& - \sum_{u=1}^a [2u]_{q_i} k_i [\dots [[\dots [X_{i,\tilde{d}_i}^+, X_{i,0}^+]_{q_i^2}, \dots X_{i,0}^+]_{q_i^{2(u-1)}}, X_{i,0}^+]_{q_i^{2u}}, \dots X_{i,0}^+]_{q_i^{2(a-1)}} = \\
& = (b_{i\tilde{d}_i} - [a]_{q_i} [a+1]_{q_i}) k_i Y_{i,a-1};
\end{aligned}$$

$$\begin{aligned}
& \text{ii) } [X_{i,j;1-a_{ij};1}^+(0;0), X_{j,\tilde{d}_{ij}}^-] = \\
& = [[\dots [X_{j,0}^+, X_{i,0}^+]_{q_i^{a_{ij}}}, \dots X_{i,0}^+]_{q_i^{a_{ij}+2(u-1)}}, \dots X_{i,0}^+]_{q_i^{-a_{ij}}}, X_{j,\tilde{d}_{ij}}^-] = \\
& = \frac{1}{q_j - q_j^{-1}} [\dots [C^{-\tilde{d}_{ij}} k_j \tilde{H}_{j,\tilde{d}_{ij}}^+, X_{i,0}^+]_{q_i^{a_{ij}}}, \dots X_{i,0}^+]_{q_i^{a_{ij}+2(u-1)}}, \dots X_{i,0}^+]_{q_i^{-a_{ij}}} = \\
& = \frac{C^{-\tilde{d}_{ij}} k_j}{q_j - q_j^{-1}} [\dots [\tilde{H}_{j,\tilde{d}_{ij}}^+, X_{i,0}^+]_{q_i^{2(u-1)}}, \dots X_{i,0}^+]_{q_i^{-2a_{ij}}}.
\end{aligned}$$

Recalling remark §7.18 we get

$$\begin{aligned}
& [X_{i,j;1-a_{ij};1}^+(0;0), X_{j,\tilde{d}_{ij}}^-] = \\
& = C^{-\tilde{d}_{ij}} k_j b_{j\tilde{d}_{ij}} [\dots [X_{i,\tilde{d}_{ij}}^+, X_{i,0}^+]_{q_i^2}, \dots X_{i,0}^+]_{q_i^{2(u-1)}}, \dots X_{i,0}^+]_{q_i^{-2a_{ij}}} = \\
& = C^{-\tilde{d}_{ij}} k_j b_{j\tilde{d}_{ij}} Y_{i,-a_{ij}}.
\end{aligned}$$

Corollary 6.

Let $i, j \in I_0$ be such that $a_{ij} < 0$ with the condition that $a_{ij} = -1$ if $k > 1$; then $M_i^+(0) \in \mathcal{I}_{cte}^+(SUL)$.

In particular:

- i) in the cases of rank higher than 1 (that is $X_{\tilde{n}}^{(k)} \neq A_1^{(1)}, A_2^{(2)}$) and different from $D_{n+1}^{(2)}$ and $D_4^{(3)}$ we have $\mathcal{I}_{cte}^\pm(X1) \subseteq \mathcal{I}_{cte}^\pm(SUL)$;
- ii) in the cases $D_{n+1}^{(2)}$ and $D_4^{(3)}$ we have $\mathcal{I}_{cte}^\pm(X1) \subseteq (\mathcal{I}_{cte}^\pm(SUL), \mathcal{I}_{cte}^\pm(X1_1))$.

Proof: This is an immediate consequence of lemma 5 (and of corollary §9.5) once one notices that the hypotheses imply that $X_{i,j;1-a_{ij};1}^+(0;0) \in \mathcal{I}_{cte}^+(SUL)$, $b_{j\tilde{d}_{ij}} \neq 0$ and $b_{i\tilde{d}_i} = [2]_{q_i}$.

Remark 7.

If $k = 2$, $X_{\tilde{n}}^{(k)} \neq A_{2n}^{(2)}$ and $i, j \in I_0$ are such that $a_{ij} = -2$ then

$$\mathcal{I}_{cte}^\pm(X1_i) \subseteq \mathcal{I}_{cte}^\pm(T2).$$

In particular $\mathcal{I}_{cte}^\pm(T2) = \mathcal{I}_{cte}^\pm(X1_i, S2)$.

Proof: $[[X_{j,0}^+, X_{i,1}^+]_{q^2}, X_{i,0}^+]$ lies in $\mathcal{I}_{cte}^+(T2)$ and so does

$$\begin{aligned}
& [[[X_{j,0}^+, X_{i,1}^+]_{q^2}, X_{i,0}^+], X_{j,0}^-] = \\
& = \left[\left[\frac{k_j - k_j^{-1}}{q^2 - q^{-2}}, X_{i,1}^+ \right]_{q^2}, X_{i,0}^+ \right] =
\end{aligned}$$

$$= -q^2 k_j [X_{i,1}^+, X_{i,0}^+]_{q^2}.$$

Theorem 8.

$$\begin{aligned} \text{i) } \mathcal{U}_q^{Dr}(X_{\tilde{n}}^{(1)}) &= \begin{cases} \tilde{\mathcal{U}}_q^{Dr}(A_1^{(1)})/\mathcal{I}_{cte}(X1) & \text{if } X_{\tilde{n}} = A_1 \\ \tilde{\mathcal{U}}_q^{Dr}(X_{\tilde{n}}^{(1)})/\mathcal{I}_{cte}(SUL) & \text{otherwise;} \end{cases} \\ \text{ii) } \mathcal{U}_q^{Dr}(X_{\tilde{n}}^{(2)}) &= \begin{cases} \tilde{\mathcal{U}}_q^{Dr}(A_{2n}^{(2)})/\mathcal{I}_{cte}(X3^1, SUL, S2) & \text{if } X_{\tilde{n}} = A_{2n} \\ \tilde{\mathcal{U}}_q^{Dr}(X_{\tilde{n}}^{(2)})/\mathcal{I}_{cte}(SUL, T2) & \text{otherwise;} \end{cases} \\ \text{iii) } \mathcal{U}_q^{Dr}(D_4^{(3)}) &= \tilde{\mathcal{U}}_q^{Dr}(D_4^{(3)})/\mathcal{I}_{cte}(X1_1, SUL, S3) = \tilde{\mathcal{U}}_q^{Dr}(D_4^{(3)})/\mathcal{I}_{cte}(X1_1, SUL, T3). \end{aligned}$$

Corollary 9.

Let: U be a $\mathbb{C}(q)$ -algebra, $t_i^{(U)}$ ($i \in I_0$) be $\mathbb{C}(q)$ -automorphisms of U , $\Omega^{(U)}$ be a \mathbb{C} -anti-linear anti-automorphism of U , $\tilde{f} : \tilde{\mathcal{U}}_q^{Dr}(X_{\tilde{n}}^{(k)}) \rightarrow U$ be a homomorphism of $\mathbb{C}(q)$ -algebras such that $\tilde{f} \circ \tilde{t}_i = t_i^{(U)} \circ \tilde{f} \forall i \in I_0$ and $\tilde{f} \circ \tilde{\Omega} = \Omega^{(U)} \circ \tilde{f}$.

If:

- i) $\tilde{f}(\mathcal{I}_0^+(X1)) = 0$ in case $X_{\tilde{n}}^{(k)} = A_1^{(1)}$;
- ii) $\tilde{f}(\mathcal{I}_0^+(SUL)) = 0$ in case $k = 1$, $X_{\tilde{n}}^{(k)} \neq A_1^{(1)}$;
- iii) $\tilde{f}(\mathcal{I}_0^+(X3^1, SUL, S2)) = 0$ in case $X_{\tilde{n}}^{(k)} = A_{2n}^{(2)}$;
- iv) $\tilde{f}(\mathcal{I}_0^+(SUL, T2)) = 0$ in case $k = 2$, $X_{\tilde{n}}^{(k)} \neq A_{2n}^{(2)}$;
- v) $\tilde{f}(\mathcal{I}_0^+(X1_1, SUL, T3)) = 0$ in case $D_4^{(3)}$;

then:

\tilde{f} induces $f : \mathcal{U}_q^{Dr}(X_{\tilde{n}}^{(k)}) \rightarrow U$ and $f \circ t_i = t_i^{(U)} \circ f \forall i \in I_0$, $f \circ \Omega = \Omega^{(U)} \circ f$.

Proof: Since the hypotheses imply that $\ker(\tilde{f})$ is a t_i -stable ($\forall i \in I_0$) and $\tilde{\Omega}$ -stable ideal of $\mathcal{U}_q^{Dr}(X_{\tilde{n}}^{(k)})$, the claim is an immediate consequence of theorem 9 and of corollary §9.5,i).

§11. The SERRE RELATIONS.

This section is devoted to the study of the Serre relations (see definition §5.9). In particular we prove that the Serre relations hold in \mathcal{U}_q^{Dr} , and that in the case of rank higher than 1 the Serre relations alone are indeed equivalent to (XD^\pm) -($S3^\pm$) (in $\tilde{\mathcal{U}}_q^{Dr}$), that is $\mathcal{U}_q^{Dr} = \tilde{\mathcal{U}}_q^{Dr}/\mathcal{I}_{cte}(S^\pm)$. We use the notations fixed in notation §5.1 and §9.1.

Remark 1.

- i) If $k = 1$ (S^\pm) = (SUL^\pm);

ii) if $k > 1$ and $i, j \in I_0$ are such that $a_{ij} < -1$ then $(S^\pm) = (SUL^\pm) \cup (S_{(i,j)}^\pm)$.

Before passing to prove that the Serre relations hold in \mathcal{U}_q^{Dr} , we state the following remark on q -commutators, which simplifies many computations in the next propositions.

Remark 2.

Let $a \in \tilde{\mathcal{U}}_q^{Dr}$, $i \in I_0$ such that $(X_n^{(k)}, i) \neq (A_{2n}^{(2)}, 1)$, $u, v \in \mathbb{C}(q)$. Then in $\tilde{\mathcal{U}}_q^{Dr}/\mathcal{I}_{cte}^+(X1_i)$ we have, for all $r \in \mathbb{Z}$:

- i) $[[a, X_{i,r}^+]_u, X_{i,r+\tilde{d}_i}^+]_v = q_i^{-2}[[a, X_{i,r+\tilde{d}_i}^+]_{q_i^2 v}, X_{i,r}^+]_{q_i^2 u}$;
- ii) $[[a, X_{i,r+2\tilde{d}_i}^+]_u, X_{i,r}^+]_v = q_i^2[[a, X_{i,r}^+]_{q_i^{-2} v}, X_{i,r+2\tilde{d}_i}^+]_{q_i^{-2} u} + (q^2 - 1)[a, (X_{i,r+\tilde{d}_i}^+)^2]_{q_i^{-2} uv}$.

Proof: It is a simple computation using remark §4.17,iii).

Proposition 3.

If $k = 2$, $X_n^{(2)} \neq A_{2n}^{(2)}$ and $i, j \in I_0$ are such that $a_{ij} = -2$, then

$$\mathcal{I}_{cte}^\pm(S_{(i,j)}) \subseteq \mathcal{I}_{cte}^\pm(T2).$$

Proof: In this proof we use that $\mathcal{I}_{cte}^\pm(X1_i) \subseteq \mathcal{I}_{cte}^\pm(T2)$ (see proposition §10.7) and make the following computations in $\tilde{\mathcal{U}}_q^{Dr}/\mathcal{I}_{cte}^+(T2)$.

Since $[H_{i,1}, [[X_{j,0}^+, X_{i,0}^+]_{q^2}, X_{i,-1}^+]]$ lies in $\mathcal{I}_{cte}^+(T2)$ (see definition §5.7) we see that

$$[[X_{j,0}^+, X_{i,1}^+]_{q^2}, X_{i,-1}^+] + [[X_{j,0}^+, X_{i,0}^+]_{q^2}, X_{i,0}^+] = 0$$

in $\tilde{\mathcal{U}}_q^{Dr}/\mathcal{I}_{cte}^+(T2)$; but, thanks to lemma 2,ii), we have that

$$[[X_{j,0}^+, X_{i,1}^+]_{q^2}, X_{i,-1}^+] = q^2[[X_{j,0}^+, X_{i,-1}^+]_{q^{-2}}, X_{i,1}^+] + (q^2 - 1)[X_{j,0}^+, (X_{i,0}^+)^2],$$

so that

$$\begin{aligned} 0 &= q^2[[X_{j,0}^+, X_{i,-1}^+]_{q^{-2}}, X_{i,1}^+] + [X_{j,0}^+, (q^2 - 1)(X_{i,0}^+)^2] + [[X_{j,0}^+, X_{i,0}^+]_{q^2}, X_{i,0}^+] = \\ &= q^2[[X_{j,0}^+, X_{i,-1}^+]_{q^{-2}}, X_{i,1}^+] + q^2[[X_{j,0}^+, X_{i,0}^+]_{q^{-2}}, X_{i,0}^+] \end{aligned}$$

and also

$$[[[X_{j,0}^+, X_{i,-1}^+]_{q^{-2}}, X_{i,1}^+] + [[X_{j,0}^+, X_{i,0}^+]_{q^{-2}}, X_{i,0}^+], X_{i,0}^+]_{q^2} = 0.$$

Now, thanks to remark 2,i) and to relations $(T2^+)$,

$$[[[X_{j,0}^+, X_{i,-1}^+]_{q^{-2}}, X_{i,1}^+], X_{i,0}^+]_{q^2} = [[X_{j,0}^+, X_{i,0}^+]_{q^2}, X_{i,-1}^+], X_{i,1}^+]_{q^{-2}} = 0,$$

so that

$$[[[X_{j,0}^+, X_{i,0}^+]_{q^{-2}}, X_{i,0}^+], X_{i,0}^+]_{q^2} = 0,$$

which implies $\mathcal{I}_{cte}^\pm(S_{(i,j)}) \subseteq \mathcal{I}_{cte}^\pm(T2)$, thanks to corollary §9.5.

Let us concentrate now on the case $A_{2n}^{(2)}$.

Lemma 4.

Let $X_{\tilde{n}}^{(k)} = A_{2n}^{(2)}$; then

$$[[X_{1,2}^+, X_{1,1}^+]_{q^2}, X_{1,0}^+]_{q^4} - (q^2 - 1)(q^4 - 1)(q^2 + q^{-2})(X_{1,1}^+)^3 \in \mathcal{I}_{cte}^+(X3^1).$$

Proof: By corollary §10.2

$$[[X_{1,2}^+, X_{1,1}^+]_{q^2}, X_{1,0}^+]_{q^4} + [[X_{1,2}^+, X_{1,1}^+]_{q^2}, X_{1,1}^+]_{q^4} + [X_{1,1}^+, X_{1,1}^+]_{q^2}, X_{1,1}^+]_{q^4}$$

belongs to $\mathcal{I}^+(X3^1) = \mathcal{I}_{cte}^+(X3^1)$.

But $[X_{1,2}^+, X_{1,0}^+]_{q^2} - (q^4 - q^{-2})(X_{1,1}^+)^2 \in \mathcal{I}_{cte}^+(X3^1)$ (see proposition §10.1), so that

$$[[X_{1,2}^+, X_{1,1}^+]_{q^2}, X_{1,0}^+]_{q^4} + (1 - q^4)(q^4 - q^{-2} + 1 - q^2)(X_{1,1}^+)^3$$

lies in $\mathcal{I}_{cte}^+(X3^1)$.

Proposition 5.

If $X_{\tilde{n}}^{(2)} = A_{2n}^{(2)}$ and $i, j \in I_0$ are such that $a_{ij} = -2$ ($i = 1, j = 2$), then $\mathcal{I}_{cte}^\pm(S_{(i,j)}) \subseteq \mathcal{I}_{cte}^\pm(XD, X3^1, S2)$.

Proof: Recall that by the very definition of $\mathcal{I}_{cte}^+(S2)$ we have (see remark §5.6)

$$(q^2 + q^{-2})[[X_{j,0}^+, X_{i,1}^+]_{q^2}, X_{i,0}^+] + q^2[[X_{i,1}^+, X_{i,0}^+]_{q^2}, X_{j,0}^+]_{q^{-4}} \in \mathcal{I}_{cte}^+(S2)$$

so that also

$$(q^2 + q^{-2})[X_{i,-1}^+, [[X_{j,0}^+, X_{i,1}^+]_{q^2}, X_{i,0}^+]]_{q^{-2}} + q^2[X_{i,-1}^+, [[X_{i,1}^+, X_{i,0}^+]_{q^2}, X_{j,0}^+]_{q^{-4}}]_{q^{-2}}$$

belongs to $\mathcal{I}_{cte}^+(S2)$, and let us compute the two summands separately in the algebra $\mathcal{U}_q^{Dr}/\mathcal{I}_{cte}^+(XD, X3^1, S2)$:

$$\begin{aligned} & [X_{i,-1}^+, [[X_{j,0}^+, X_{i,1}^+]_{q^2}, X_{i,0}^+]]_{q^{-2}} = \\ & = -q^2[X_{i,-1}^+, [[X_{i,1}^+, X_{j,0}^+]_{q^{-2}}, X_{i,0}^+]]_{q^{-2}} \stackrel{XD}{=} q^2[X_{i,-1}^+, [[X_{j,1}^+, X_{i,0}^+]_{q^{-2}}, X_{i,0}^+]]_{q^{-2}} = \\ & = q^2[[X_{i,-1}^+, [X_{j,1}^+, X_{i,0}^+]_{q^{-2}}], X_{i,0}^+]_{q^{-2}} + q^2[[X_{j,1}^+, X_{i,0}^+]_{q^{-2}}, [X_{i,-1}^+, X_{i,0}^+]_{q^{-2}}] \stackrel{S2}{=} \\ & \stackrel{S2}{=} -q^4[[X_{i,0}^+, [X_{j,1}^+, X_{i,-1}^+]_{q^{-2}}]_{q^{-4}}, X_{i,0}^+]_{q^{-2}} - [[X_{j,1}^+, X_{i,0}^+]_{q^{-2}}, [X_{i,0}^+, X_{i,-1}^+]_{q^2}] = \\ & = -q^4[[X_{i,0}^+, [X_{j,1}^+, X_{i,-1}^+]_{q^{-2}}]_{q^{-4}}, X_{i,0}^+]_{q^{-2}} + \\ & - [X_{j,1}^+, [X_{i,0}^+, [X_{i,0}^+, X_{i,-1}^+]_{q^2}]_{q^4}]_{q^{-6}} + q^{-2}[X_{i,0}^+, [X_{j,1}^+, [X_{i,0}^+, X_{i,-1}^+]_{q^2}]_{q^{-4}}]_{q^6} \stackrel{XD, X3^{-1}}{=} \\ & \stackrel{XD, X3^{-1}}{=} q^4[[X_{i,0}^+, [X_{i,0}^+, X_{j,0}^+]_{q^{-2}}]_{q^{-4}}, X_{i,0}^+]_{q^{-2}} + q^{-2}[X_{i,0}^+, [X_{j,1}^+, [X_{i,0}^+, X_{i,-1}^+]_{q^2}]_{q^{-4}}]_{q^6} \stackrel{S2}{=} \\ & \stackrel{S2}{=} q^{-2}[[[X_{j,0}^+, X_{i,0}^+]_{q^2}, X_{i,0}^+]_{q^4}, X_{i,0}^+]_{q^{-2}} - (1 + q^{-4})[X_{i,0}^+, [[X_{j,1}^+, X_{i,-1}^+]_{q^{-2}}, X_{i,0}^+]_{q^6}] \stackrel{XD}{=} \\ & \stackrel{XD}{=} q^{-2}[[[X_{j,0}^+, X_{i,0}^+]_{q^2}, X_{i,0}^+]_{q^4}, X_{i,0}^+]_{q^{-2}} + (q^4 + 1)[[X_{j,0}^+, X_{i,0}^+]_{q^2}, X_{i,0}^+]_{q^{-6}} \end{aligned}$$

and

$$\begin{aligned}
& [X_{i,-1}^+, [[X_{i,1}^+, X_{i,0}^+]_{q^2}, X_{j,0}^+]_{q^{-4}}]_{q^{-2}} = \\
& = [[X_{i,-1}^+, [X_{i,1}^+, X_{i,0}^+]_{q^2}]_{q^{-4}}, X_{j,0}^+]_{q^{-2}} + q^{-4}[[X_{i,1}^+, X_{i,0}^+]_{q^2}, [X_{i,-1}^+, X_{j,0}^+]_{q^2}] \stackrel{XD}{=} \\
& \stackrel{XD}{=} q^{-6}[X_{j,0}^+, [[X_{i,1}^+, X_{i,0}^+]_{q^2}, X_{i,-1}^+]_{q^4}]_{q^2} + q^{-2}[[X_{i,1}^+, X_{i,0}^+]_{q^2}, [X_{i,0}^+, X_{j,-1}^+]_{q^{-2}}] \stackrel{X3^1}{=} \\
& \stackrel{X3^1}{=} (1-q^{-2})(1-q^{-4})(q^2+q^{-2})[X_{j,0}^+, (X_{i,0}^+)^3]_{q^2} - q^{-2}[[X_{i,0}^+, X_{j,-1}^+]_{q^{-2}}, [X_{i,1}^+, X_{i,0}^+]_{q^2}] = \\
& = (1-q^{-2})(1-q^{-4})(q^2+q^{-2})[X_{j,0}^+, (X_{i,0}^+)^3]_{q^2} + \\
& -q^{-2}[X_{i,0}^+, [X_{j,-1}^+, [X_{i,1}^+, X_{i,0}^+]_{q^2}]_{q^4}]_{q^{-6}} + q^{-4}[X_{j,-1}^+, [X_{i,0}^+, [X_{i,1}^+, X_{i,0}^+]_{q^2}]_{q^{-4}}]_{q^6} \stackrel{X3^1}{=} \\
& \stackrel{X3^1}{=} (1-q^{-2})(1-q^{-4})(q^2+q^{-2})[X_{j,0}^+, (X_{i,0}^+)^3]_{q^2} + q^2[X_{i,0}^+, [[X_{i,1}^+, X_{i,0}^+]_{q^2}, X_{j,-1}^+]_{q^{-4}}]_{q^{-6}} \stackrel{S2}{=} \\
& \stackrel{S2}{=} (q^2+q^{-2})((1-q^{-2})(1-q^{-4})[X_{j,0}^+, (X_{i,0}^+)^3]_{q^2} - [X_{i,0}^+, [[X_{j,-1}^+, X_{i,1}^+]_{q^2}, X_{i,0}^+]_{q^{-6}}]) \stackrel{XD}{=} \\
& \stackrel{XD}{=} (q^2+q^{-2})((1-q^{-2})(1-q^{-4})[X_{j,0}^+, (X_{i,0}^+)^3]_{q^2} + q^{-4}[[X_{j,0}^+, X_{i,0}^+]_{q^{-2}}, X_{i,0}^+]_{q^6}).
\end{aligned}$$

It follows that

$$\begin{aligned}
& q^{-2}[[X_{j,0}^+, X_{i,0}^+]_{q^2}, X_{i,0}^+]_{q^4}, X_{i,0}^+]_{q^{-2}} + (q^4+1)[[[X_{j,0}^+, X_{i,0}^+]_{q^2}, X_{i,0}^+], X_{i,0}^+]_{q^{-6}} + \\
& + (q^2-1)(1-q^{-4})[X_{j,0}^+, (X_{i,0}^+)^3]_{q^2} + q^{-2}[[X_{j,0}^+, X_{i,0}^+]_{q^{-2}}, X_{i,0}^+]_{q^6} = \\
& = (q+q^{-1})(q^3+q^{-3})(X_{j,0}^+(X_{i,0}^+)^3 - (q^2+1+q^{-2})X_{i,0}^+X_{j,0}^+(X_{i,0}^+)^2 + \\
& + (q^2+1+q^{-2})(X_{i,0}^+)^2X_{j,0}^+X_{i,0}^+ - (X_{i,0}^+)^3X_{j,0}^+)
\end{aligned}$$

is an element of $\mathcal{I}_{cte}^+(XD, X3^1, S2)$, hence $\mathcal{I}_0^+(S_{(i,j)}) \subseteq \mathcal{I}_{cte}^+(XD, X3^1, S2)$.

Thanks to corollary §9.5 we obtain $\mathcal{I}_{cte}^\pm(S_{(i,j)}) \subseteq \mathcal{I}_{cte}^\pm(XD, X3^1, S2)$.

Proposition 6.

If $k = 3$ and $i, j \in I_0$ are such that $a_{ij} = -3$, then $\mathcal{I}_{cte}^\pm(S_{(i,j)}) \subseteq \mathcal{I}_{cte}^\pm(X1_i, T3)$.

Proof: Let us start from

$$(q^2+1)[[X_{j,0}^+, X_{i,2}^+]_{q^3}, X_{i,0}^+]_{q^{-1}} + [[X_{j,0}^+, X_{i,1}^+]_{q^3}, X_{i,1}^+]_q,$$

which is an element of $\mathcal{I}_{cte}^+(T3)$, and remark that

$$[[X_{j,0}^+, X_{i,2}^+]_{q^3}, X_{i,0}^+]_{q^{-1}} - q^2[[X_{j,0}^+, X_{i,0}^+]_{q^{-3}}, X_{i,2}^+]_q - (q^2-1)[X_{j,0}^+, (X_{i,1}^+)^2]$$

belongs to $\mathcal{I}_{cte}^+(X1_i)$; but

$$\begin{aligned}
& (q^2+1)(q^2-1)[X_{j,0}^+, (X_{i,1}^+)^2] + [[X_{j,0}^+, X_{i,1}^+]_{q^3}, X_{i,1}^+]_q = \\
& = q^4[[X_{j,0}^+, X_{i,1}^+]_{q^{-3}}, X_{i,1}^+]_{q^{-1}},
\end{aligned}$$

so that

$$[[X_{j,0}^+, X_{i,1}^+]_{q^{-3}}, X_{i,1}^+]_{q^{-1}} + (1+q^{-2})[[X_{j,0}^+, X_{i,0}^+]_{q^{-3}}, X_{i,2}^+]_q$$

lies in $\mathcal{I}_{cte}^+(X1_i, T3)$, hence

$$[[X_{j,-3}^+, X_{i,4}^+]_{q^3}, X_{i,1}^+]_{q^{-1}} + (1+q^{-2})[[X_{j,-3}^+, X_{i,3}^+]_{q^3}, X_{i,2}^+]_q$$

and (applying $\tilde{t}_j^{-1}\tilde{t}_i^2$ and q -commuting by $X_{i,0}^+$)

$$[[[X_{j,0}^+, X_{i,2}^+]_{q^3}, X_{i,-1}^+]_{q^{-1}} + (1 + q^{-2})[[X_{j,0}^+, X_{i,1}^+]_{q^3}, X_{i,0}^+]_q, X_{i,0}^+]_{q^{-3}}$$

lie in $\mathcal{I}_{cte}^+(XD, X1_i, T3)$; but

$$[[[X_{j,0}^+, X_{i,2}^+]_{q^3}, X_{i,-1}^+]_{q^{-1}}, X_{i,0}^+]_{q^{-3}} - q^{-2}[[[X_{j,0}^+, X_{i,2}^+]_{q^3}, X_{i,0}^+]_{q^{-1}}, X_{i,-1}^+]_q$$

belongs to $\mathcal{I}_{cte}^+(X1_i)$ by lemma 2,

$$q^{-2}[[[X_{j,0}^+, X_{i,2}^+]_{q^3}, X_{i,0}^+]_{q^{-1}}, X_{i,-1}^+]_q + \frac{q^{-2}}{q^2 + 1}[[[X_{j,0}^+, X_{i,1}^+]_{q^3}, X_{i,1}^+]_q, X_{i,-1}^+]_q$$

belongs to $\mathcal{I}_{cte}^+(T3)$,

$$[[[X_{j,0}^+, X_{i,1}^+]_{q^3}, X_{i,1}^+]_q, X_{i,-1}^+]_q +$$

$$-q^2[[[X_{j,0}^+, X_{i,1}^+]_{q^3}, X_{i,-1}^+]_{q^{-1}}, X_{i,1}^+]_{q^{-1}} - (q^2 - 1)[[X_{j,0}^+, X_{i,1}^+]_{q^3}, (X_{i,0}^+)^2]$$

belongs to $\mathcal{I}_{cte}^+(X1_i)$ again by lemma 2,

$$q^2[[[X_{j,0}^+, X_{i,1}^+]_{q^3}, X_{i,-1}^+]_{q^{-1}}, X_{i,1}^+]_{q^{-1}} + \frac{q^2}{q^2 + 1}[[[X_{j,0}^+, X_{i,0}^+]_{q^3}, X_{i,0}^+]_q, X_{i,1}^+]_{q^{-1}}$$

belongs to $\mathcal{I}_{cte}^+(T3)$ and

$$[[[X_{j,0}^+, X_{i,0}^+]_{q^3}, X_{i,0}^+]_q, X_{i,1}^+]_{q^{-1}} - q^{-4}[[[X_{j,0}^+, X_{i,1}^+]_{q^3}, X_{i,0}^+]_{q^5}, X_{i,0}^+]_{q^3}$$

belongs to $\mathcal{I}_{cte}^+(X1_i)$ (by lemma 2). So we can conclude that

$$\begin{aligned} & \frac{q^{-4}}{(q^2 + 1)^2}[[[X_{j,0}^+, X_{i,1}^+]_{q^3}, X_{i,0}^+]_{q^5}, X_{i,0}^+]_{q^3} - \frac{1 - q^{-2}}{q^2 + 1}[[X_{j,0}^+, X_{i,1}^+]_{q^3}, (X_{i,0}^+)^2] + \\ & + (1 + q^{-2})[[[X_{j,0}^+, X_{i,1}^+]_{q^3}, X_{i,0}^+]_q, X_{i,0}^+]_{q^{-3}} = \\ & = \frac{(q^2 + 1 + q^{-2})^2}{(q^2 + 1)^2}[[[X_{j,0}^+, X_{i,1}^+]_{q^3}, X_{i,0}^+]_q, X_{i,0}^+]_{q^{-1}} \end{aligned}$$

lies in $\mathcal{I}_{cte}^+(XD, X1_i, T3)$.

Then $[[[X_{j,0}^+, X_{i,1}^+]_{q^3}, X_{i,0}^+]_q, X_{i,0}^+]_{q^{-1}}, X_{i,-1}^+]_{q^3} \in \mathcal{I}_{cte}^+(X1_i, T3)$; but

$$[[[X_{j,0}^+, X_{i,1}^+]_{q^3}, X_{i,0}^+]_q, X_{i,0}^+]_{q^{-1}}, X_{i,-1}^+]_{q^3} +$$

$$-q^4[[[X_{j,0}^+, X_{i,1}^+]_{q^3}, X_{i,-1}^+]_{q^{-1}}, X_{i,0}^+]_{q^{-1}}, X_{i,0}^+]_{q^{-3}}$$

belongs to $\mathcal{I}_{cte}^\pm(X1_i)$ and since

$$(q^2 + 1)[[X_{j,0}^+, X_{i,1}^+]_{q^3}, X_{i,-1}^+]_{q^{-1}} + [[X_{j,0}^+, X_{i,0}^+]_{q^3}, X_{i,0}^+]_q \in \mathcal{I}_{cte}^+(T3)$$

it follows that

$$[[[X_{j,0}^+, X_{i,0}^+]_{q^3}, X_{i,0}^+]_q, X_{i,0}^+]_{q^{-1}}, X_{i,0}^+]_{q^{-3}} \in \mathcal{I}_{cte}^+(X1, T3),$$

so that $\mathcal{I}_0^+(S_{(i,j)}) \subseteq \mathcal{I}_{cte}^+(X1, T3)$. Then $\mathcal{I}_{cte}^\pm(S_{(i,j)}) \subseteq \mathcal{I}_{cte}^\pm(X1, T3)$.

Corollary 7.

$\mathcal{I}_{cte}(S) = 0$ in \mathcal{U}_q^{Dr} .

Proof: The claim is a straightforward consequence of remark 1 and propositions 3, 5, 6.

We are now able to prove that the quantum algebra of finite type \mathcal{U}_q^{fin} is mapped in \mathcal{U}_q^{Dr} , which was not otherwise clear.

Definition 8.

Let $\phi : \mathcal{U}_q^{fin} \rightarrow \mathcal{U}_q^{Dr}$ be the $\mathbb{C}(q)$ -homomorphism given by

$$K_i^{\pm 1} \mapsto k_i^{\pm 1}, \quad E_i \mapsto X_{i,0}^+, \quad F_i \mapsto X_{i,0}^- \quad (i \in I_0).$$

Remark 9.

ϕ is well defined.

Proof: It is a straightforward consequence of corollary 7 (and of relations (CUK) , (CK) , (KX^\pm) , (XXE)).

We shall now complete the study of the ideal generated by the Serre relations.

Remark 10.

$$\mathcal{I}_{cte}^\pm(XD) \subseteq \mathcal{I}_{cte}^\pm(S).$$

$$\text{If } n > 1 \quad \mathcal{I}_{cte}^\pm(X1), \mathcal{I}_{cte}^\pm(X2), \mathcal{I}_{cte}^\pm(X3^1), \mathcal{I}_{cte}^\pm(X3^{-1}) \subseteq \mathcal{I}_{cte}^\pm(S).$$

Proof: That $\mathcal{I}_{cte}^\pm(XD) \subseteq \mathcal{I}_{cte}^\pm(S)$ follows from proposition §10.4 and from remark 1.

That $\mathcal{I}_{cte}^\pm(X1) \subseteq \mathcal{I}_{cte}^\pm(S)$ is a consequence of lemma §10.5 and of corollary §9.5 (see also corollary §10.6).

Finally that $\mathcal{I}_{cte}^\pm(X3^1) \subseteq \mathcal{I}_{cte}^\pm(S)$ follows again from lemma §10.5 and from corollary §9.5, once one notices that $(X_{\tilde{n}}^{(k)}, i, j) = (A_{2n}^{(2)}, 1, 2)$ implies $q_i = q$, $b_{j\tilde{i}d_{ij}} \neq 0$ and $b_{i\tilde{i}d_i} = [2]_q[3]_q$.

From this it follows that $\mathcal{I}_{cte}^\pm(X2), \mathcal{I}_{cte}^\pm(X3^{-1}) \subseteq \mathcal{I}_{cte}^\pm(S)$ (see proposition §10.1).

Corollary 11.

- i) $\mathcal{I}^\pm(S) = \mathcal{I}_{cte}^\pm(S)$ (see corollary §9.17,iii) and remark 10);
- ii) $\mathcal{I}(S) = 0$ in \mathcal{U}_q^{Dr} (see corollary 7).

Remark 12.

If $k > 1$ and $X_{\tilde{n}} \neq A_{2n}$ we have that $\mathcal{I}_{cte}^\pm(Sk) \subseteq \mathcal{I}_{cte}^\pm(S) \Leftrightarrow \mathcal{I}_{cte}^\pm(Tk) \subseteq \mathcal{I}_{cte}^\pm(S)$.

Proof: Of course we can suppose $n > 1$; then the claim depends on the fact that $(\mathcal{I}_{cte}^\pm(X1), \mathcal{I}_{cte}^\pm(Sk)) = (\mathcal{I}_{cte}^\pm(X1), \mathcal{I}_{cte}^\pm(Tk))$ (see remark §5.8) and that $\mathcal{I}_{cte}^\pm(X1) \subseteq \mathcal{I}_{cte}^\pm(S)$ (see remark 10).

Proposition 13.

$$\mathcal{I}_{cte}^\pm(S2) \subseteq \mathcal{I}_{cte}^\pm(S).$$

Proof: Let $k = 2$ and $i, j \in I_0$ be such that $a_{ij} = -2$.

Then $[[[X_{j,0}^+, X_{i,0}^+]_{q^2}, X_{i,0}^+], X_{i,0}^+]_{q^{-2}}$ is an element of $\mathcal{I}_{cte}^+(S)$, so that

$$\begin{aligned} & \mathcal{I}_{cte}^+(S) \ni [[[[X_{j,0}^+, X_{i,0}^+]_{q^2}, X_{i,0}^+], X_{i,0}^+]_{q^{-2}}, X_{i,1}^-] = \\ &= [[[[X_{j,0}^+, C^{-1}k_i H_{i,1}]_{q^2}, X_{i,0}^+], X_{i,0}^+]_{q^{-2}} + [[[[X_{j,0}^+, X_{i,0}^+]_{q^2}, C^{-1}k_i H_{i,1}], X_{i,0}^+]_{q^{-2}} + \\ & \quad + [[[[X_{j,0}^+, X_{i,0}^+]_{q^2}, X_{i,0}^+], C^{-1}k_i H_{i,1}]_{q^{-2}}] = \\ &= -C^{-1}k_i(q^2[[[H_{i,1}, X_{j,0}^+], X_{i,0}^+]_{q^{-2}}, X_{i,0}^+]_{q^{-4}} + [[[[H_{i,1}, X_{j,0}^+], X_{i,0}^+]_{q^2}, X_{i,0}^+]_{q^{-4}} + \\ & \quad + [[X_{j,0}^+, [H_{i,1}, X_{i,0}^+]_{q^2}, X_{i,0}^+]_{q^{-4}} + q^{-2}[[[H_{i,1}, X_{j,0}^+], X_{i,0}^+]_{q^2}, X_{i,0}^+] + \\ & \quad + q^{-2}[[X_{j,0}^+, [H_{i,1}, X_{i,0}^+]_{q^2}, X_{i,0}^+] + q^{-2}[[X_{j,0}^+, X_{i,0}^+]_{q^2}, [H_{i,1}, X_{i,0}^+]]) = \\ &= -C^{-1}k_i(b_{ij1}(q^2[[X_{j,1}^+, X_{i,0}^+]_{q^{-2}}, X_{i,0}^+]_{q^{-4}} + [[X_{j,1}^+, X_{i,0}^+]_{q^2}, X_{i,0}^+]_{q^{-4}} + \\ & \quad + q^{-2}[[X_{j,1}^+, X_{i,0}^+]_{q^2}, X_{i,0}^+]) + b_{ii1}([[[X_{j,0}^+, X_{i,1}^+]_{q^2}, X_{i,0}^+]_{q^{-4}} + \\ & \quad + q^{-2}[[X_{j,0}^+, X_{i,1}^+]_{q^2}, X_{i,0}^+] + q^{-2}[[X_{j,0}^+, X_{i,0}^+]_{q^2}, X_{i,1}^+])) = \\ &= -C^{-1}k_i([3]_q b_{ij1}(X_{j,1}^+(X_{i,0}^+)^2 - (q^{-2} + 1)X_{i,0}^+ X_{j,1}^+ X_{i,0}^+ + q^{-2}(X_{i,0}^+)^2 X_{j,1}^+) + \\ & \quad + [3]_q b_{ii1}(q^{-2}X_{j,0}^+ X_{i,1}^+ X_{i,0}^+ - X_{i,1}^+ X_{j,0}^+ X_{i,0}^+ - q^{-2}X_{i,0}^+ X_{j,0}^+ X_{i,1}^+ + X_{i,0}^+ X_{i,1}^+ X_{j,0}^+) + \\ & \quad + b_{ii1}(-q^{-4}X_{j,0}^+[X_{i,1}^+, X_{i,0}^+]_{q^2} + [X_{i,1}^+, X_{i,0}^+]_{q^2} X_{j,0}^+)) = \\ &= -C^{-1}k_i([3]_q b_{ij1}[[X_{j,1}^+, X_{i,0}^+]_{q^{-2}}, X_{i,0}^+] + \\ & \quad + q^{-2}[3]_q b_{ii1}[[X_{j,0}^+, X_{i,1}^+]_{q^2}, X_{i,0}^+] + b_{ii1}[[X_{i,1}^+, X_{i,0}^+]_{q^2}, X_{j,0}^+]_{q^{-4}}). \end{aligned}$$

Now let us notice that if $X_{\tilde{n}} \neq A_{2n}$ we have $b_{ii1} \neq 0$, $[X_{i,1}^+, X_{i,0}^+]_{q^2} \in \mathcal{I}_{cte}^+(S)$ (see remark 10) and $\tilde{d}_j = 2$, hence $b_{ij1} = 0$: we can conclude that $[[X_{j,0}^+, X_{i,1}^+]_{q^2}, X_{i,0}^+]$ is an element of $\mathcal{I}_{cte}^+(S)$, so that $\mathcal{I}_{cte}^+(T2) \subseteq \mathcal{I}_{cte}^+(S)$ (see corollary §9.5), which, thanks to remark 12, is equivalent to $\mathcal{I}_{cte}^+(S2) \subseteq \mathcal{I}_{cte}^+(S)$.

On the other hand, if $X_{\tilde{n}} = A_{2n}$ we have $b_{ii1} = [2]_q[3]_q$, $\tilde{d}_j = 1$, $b_{ij1} = -[2]_q$ and $[X_{j,1}^+, X_{i,0}^+]_{q^{-2}} + [X_{i,1}^+, X_{j,0}^+]_{q^{-2}} \in \mathcal{I}_{cte}^+(S)$ (see remark 10); then we have that

$$(q^2 + q^{-2})[[X_{j,0}^+, X_{i,1}^+]_{q^2}, X_{i,0}^+] + q^2[[X_{i,1}^+, X_{i,0}^+]_{q^2}, X_{j,0}^+]_{q^{-4}}$$

is an element of $\mathcal{I}_{cte}^+(S)$, that is $\mathcal{I}_{cte}^+(S2) \subseteq \mathcal{I}_{cte}^+(S)$.

In both cases using corollary §9.5 we get $\mathcal{I}_{cte}^\pm(S2) \subseteq \mathcal{I}_{cte}^\pm(S)$.

Proposition 14.

$$\mathcal{I}_{cte}^\pm(T3) \subseteq \mathcal{I}_{cte}^\pm(S).$$

Proof: Let $k = 3$ ($X_{\tilde{n}}^{(k)} = D_4^{(3)}$) and $i, j \in I_0$ be such that $a_{ij} = -3$ ($i = 1, j = 2$).

Then $[[[[X_{j,0}^+, X_{i,0}^+]_{q^3}, X_{i,0}^+]_q, X_{i,0}^+]_{q^{-1}}, X_{i,0}^+]_{q^{-3}}$ is an element of $\mathcal{I}_{cte}^+(S)$, so that, recalling that $b_{ij1} = 0$ and $b_{ii1} = [2]_q$,

$$\begin{aligned} \mathcal{I}_{cte}^+(S) \ni & [[[[[X_{j,0}^+, X_{i,0}^+]_{q^3}, X_{i,0}^+]_q, X_{i,0}^+]_{q^{-1}}, X_{i,0}^+]_{q^{-3}}, X_{i,1}^-] = \\ & = [[[[X_{j,0}^+, C^{-1}k_i H_{i,1}]_{q^3}, X_{i,0}^+]_q, X_{i,0}^+]_{q^{-1}}, X_{i,0}^+]_{q^{-3}} + \\ & + [[[[X_{j,0}^+, X_{i,0}^+]_{q^3}, C^{-1}k_i H_{i,1}]_q, X_{i,0}^+]_{q^{-1}}, X_{i,0}^+]_{q^{-3}} + \\ & + [[[[X_{j,0}^+, X_{i,0}^+]_{q^3}, X_{i,0}^+]_q, C^{-1}k_i H_{i,1}]_{q^{-1}}, X_{i,0}^+]_{q^{-3}} + \\ & + [[[[X_{j,0}^+, X_{i,0}^+]_{q^3}, X_{i,0}^+]_q, X_{i,0}^+]_{q^{-1}}, C^{-1}k_i H_{i,1}]_{q^{-3}} = \\ & = -C^{-1}k_i(q^3[[[H_{i,1}, X_{j,0}^+], X_{i,0}^+]_{q^{-1}}, X_{i,0}^+]_{q^{-3}}, X_{i,0}^+]_{q^{-5}} + \\ & + q[[[H_{i,1}, [X_{j,0}^+, X_{i,0}^+]_{q^3}], X_{i,0}^+]_{q^{-3}}, X_{i,0}^+]_{q^{-5}} + \\ & + q^{-1}[[[H_{i,1}, [[X_{j,0}^+, X_{i,0}^+]_{q^3}, X_{i,0}^+]_q], X_{i,0}^+]_{q^{-5}} + \\ & + q^{-3}[H_{i,1}, [[X_{j,0}^+, X_{i,0}^+]_{q^3}, X_{i,0}^+]_q, X_{i,0}^+]_{q^{-1}}) = \\ & = -[2]_q C^{-1}k_i(q[[[X_{j,0}^+, X_{i,1}^+]_{q^3}, X_{i,0}^+]_{q^{-3}}, X_{i,0}^+]_{q^{-5}} + q^{-1}[[[X_{j,0}^+, X_{i,1}^+]_{q^3}, X_{i,0}^+]_q, X_{i,0}^+]_{q^{-5}} + \\ & + q^{-1}[[[X_{j,0}^+, X_{i,0}^+]_{q^3}, X_{i,1}^+]_q, X_{i,0}^+]_{q^{-5}} + q^{-3}[[[X_{j,0}^+, X_{i,1}^+]_{q^3}, X_{i,0}^+]_q, X_{i,0}^+]_{q^{-1}}) + \\ & + q^{-3}[[[X_{j,0}^+, X_{i,0}^+]_{q^3}, X_{i,1}^+]_q, X_{i,0}^+]_{q^{-1}}) + q^{-3}[[[X_{j,0}^+, X_{i,0}^+]_{q^3}, X_{i,0}^+]_q, X_{i,1}^+]_{q^{-1}}); \end{aligned}$$

then, thanks to remarks 10 and 2, we have that

$$\begin{aligned} & q[[[X_{j,0}^+, X_{i,1}^+]_{q^3}, X_{i,0}^+]_{q^{-3}}, X_{i,0}^+]_{q^{-5}} + q^{-1}[[[X_{j,0}^+, X_{i,1}^+]_{q^3}, X_{i,0}^+]_q, X_{i,0}^+]_{q^{-5}} + \\ & + q^{-3}[[[X_{j,0}^+, X_{i,1}^+]_{q^3}, X_{i,0}^+]_{q^5}, X_{i,0}^+]_{q^{-5}} + q^{-3}[[[X_{j,0}^+, X_{i,1}^+]_{q^3}, X_{i,0}^+]_q, X_{i,0}^+]_{q^{-1}}) + \\ & + q^{-5}[[[X_{j,0}^+, X_{i,1}^+]_{q^3}, X_{i,0}^+]_{q^5}, X_{i,0}^+]_{q^{-1}}) + q^{-7}[[[X_{j,0}^+, X_{i,0}^+]_{q^3}, X_{i,0}^+]_{q^5}, X_{i,1}^+]_{q^3} = \\ & = q^{-3}(q^2 + q^{-2})[3]_q[[[X_{j,0}^+, X_{i,1}^+]_{q^3}, X_{i,0}^+]_q, X_{i,0}^+]_{q^{-1}} \end{aligned}$$

belongs to $\mathcal{I}_{cte}^+(S)$; then so does

$$\begin{aligned} & [[[[[X_{j,0}^+, X_{i,1}^+]_{q^3}, X_{i,0}^+]_{q^{-1}}, X_{i,0}^+]_q, X_{i,-1}^-] = \\ & = \left[\left[\left[X_{j,0}^+, \frac{Ck_i - C^{-1}k_i^{-1}}{q - q^{-1}} \right]_{q^3}, X_{i,0}^+ \right]_{q^{-1}}, X_{i,0}^+ \right]_q + \\ & + [[X_{j,0}^+, X_{i,1}^+]_{q^3}, k_i^{-1}H_{i,-1}]_{q^{-1}}, X_{i,0}^+]_q + [[X_{j,0}^+, X_{i,1}^+]_{q^3}, X_{i,0}^+]_{q^{-1}}, k_i^{-1}H_{i,-1}]_q = \\ & = k_i^{-1}([3]_q C^{-1}[[X_{j,0}^+, X_{i,0}^+]_q, X_{i,0}^+]_{q^3} + \\ & - q^{-1}[[[H_{i,-1}, [X_{j,0}^+, X_{i,1}^+]_{q^3}], X_{i,0}^+]_{q^3} - q[H_{i,-1}, [[X_{j,0}^+, X_{i,1}^+]_{q^3}, X_{i,0}^+]_{q^{-1}}]) = \\ & = C^{-1}k_i^{-1}([3]_q[[X_{j,0}^+, X_{i,0}^+]_{q^3}, X_{i,0}^+]_q - q^{-1}[2]_q[[[X_{j,0}^+, X_{i,0}^+]_{q^3}, X_{i,0}^+]_{q^3} + \\ & - q[2]_q[[X_{j,0}^+, X_{i,0}^+]_{q^3}, X_{i,0}^+]_{q^{-1}} - q[2]_q[[X_{j,0}^+, X_{i,1}^+]_{q^3}, X_{i,-1}^+]_{q^{-1}}]) = \\ & = -C^{-1}k_i^{-1}([X_{j,0}^+, X_{i,0}^+]_{q^3}, X_{i,0}^+]_q + q[2]_q[[X_{j,0}^+, X_{i,1}^+]_{q^3}, X_{i,-1}^+]_{q^{-1}}); \end{aligned}$$

hence $\mathcal{I}_0^+(T3) \subseteq \mathcal{I}_{cte}^+(S)$ and, using corollary §9.5, $\mathcal{I}_{cte}^\pm(T3) \subseteq \mathcal{I}_{cte}^\pm(S)$.

Corollary 15.

$$\mathcal{I}_{cte}^\pm(S3) \subseteq \mathcal{I}_{cte}^\pm(S).$$

Proof: It follows from remark 12 and from proposition 14.

Corollary 16.

If $n > 1$ $\mathcal{I}_{cte}^\pm(XD, X1, X2, X3^{\pm 1}, SUL, S2, S3) = \mathcal{I}_{cte}^\pm(S)$.

Proof: It follows from corollary 7, remarks 1 and 10, proposition 13 and corollary 15.

Remark 17.

In $\mathcal{U}_q^{Dr}(D_4^{(3)})$ we have $[[[X_{j,0}^+, X_{i,1}^+]_{q^3}, X_{i,0}^+]_q, X_{i,0}^+]_{q^{-1}} = 0$.

Proof: See the proof of proposition 14.

Theorem 18.

- i) $\mathcal{U}_q^{Dr}(A_1^{(1)}) = \tilde{\mathcal{U}}_q^{Dr}(A_1^{(1)})/\mathcal{I}_{cte}(X1)$;
- ii) $\mathcal{U}_q^{Dr}(A_2^{(2)}) = \tilde{\mathcal{U}}_q^{Dr}(A_2^{(2)})/\mathcal{I}_{cte}(X3^1)$;
- iii) $\mathcal{U}_q^{Dr}(X_{\tilde{n}}^{(k)}) = \tilde{\mathcal{U}}_q^{Dr}(X_{\tilde{n}}^{(k)})/\mathcal{I}_{cte}(S)$ if $n > 1$ (that is $X_{\tilde{n}}^{(k)} \neq A_1^{(1)}, A_2^{(2)}$).

Proof: The claims follow from theorem §10.8 and corollary 16.

Corollary 19.

Let: U be a $\mathbb{C}(q)$ -algebra, $t_i^{(U)}$ ($i \in I_0$) be $\mathbb{C}(q)$ -automorphisms of U , $\Omega^{(U)}$ be a \mathbb{C} -anti-linear anti-automorphism of U , $\tilde{f} : \tilde{\mathcal{U}}_q^{Dr}(X_{\tilde{n}}^{(k)}) \rightarrow U$ be a homomorphism of $\mathbb{C}(q)$ -algebras such that $\tilde{f} \circ \tilde{t}_i = t_i^{(U)} \circ \tilde{f} \forall i \in I_0$ and $\tilde{f} \circ \tilde{\Omega} = \Omega^{(U)} \circ \tilde{f}$.

If:

- i) $\tilde{f}(\mathcal{I}_0^+(X1)) = 0$ in case $X_{\tilde{n}}^{(k)} = A_1^{(1)}$;
- ii) $\tilde{f}(\mathcal{I}_0^+(X3^1)) = 0$ in case $X_{\tilde{n}}^{(k)} = A_2^{(2)}$;
- iii) $\tilde{f}(\mathcal{I}_0^+(S)) = 0$ in case $X_{\tilde{n}}^{(k)} \neq A_1^{(1)}, A_2^{(2)}$;

then:

\tilde{f} induces $f : \mathcal{U}_q^{Dr}(X_{\tilde{n}}^{(k)}) \rightarrow U$ and $f \circ t_i = t_i^{(U)} \circ f \forall i \in I_0$, $f \circ \Omega = \Omega^{(U)} \circ f$.

Proof: Since the hypotheses imply that $\ker(\tilde{f})$ is a t_i -stable ($\forall i \in I_0$), $\tilde{\Omega}$ -stable ideal of $\tilde{\mathcal{U}}_q^{Dr}(X_{\tilde{n}}^{(k)})$, the claim is an immediate consequence of theorem 18 and of corollary §9.5.

Remark 20.

It is useful to compare the results of this section with those of section §10. The simplification of the relations given in section §10 (theorem §10.8 and corollary §10.9) provides a minimal set of relations of lowest “degree” (where the degree of $X_{i_1, r_1} \cdots X_{i_h, r_h}$ is meant to be h); this minimality in degree can be often useful, in spite of the appearance more complicated of relations like $(S2^\pm)$ with respect to the simple and familiar Serre relations. On the other hand the advantage of the

Serre relations is evident in all the cases, like the application of theorem 18 and corollary 19 given in section §12, when the Serre relations play a central role: to this aim recall that the Serre relations are the minimal degree relations defining the positive part of \mathcal{U}_q^{fin} (see definition 8 and remark 9, and recall [L]).

§12. The HOMOMORPHISM ψ from \mathcal{U}_q^{Dr} to \mathcal{U}_q^{DJ} .

This section is devoted to exhibit a homomorphism $\psi : \mathcal{U}_q^{Dr} \rightarrow \mathcal{U}_q^{DJ}$ and to prove that it is surjective.

Notation 1.

In the following $o : I_0 \rightarrow \{\pm 1\}$ will be a map such that:

- a) $a_{ij} \neq 0 \Rightarrow o(i)o(j) = -1$ (see [Be] for the untwisted case);
- b) in the twisted case different from $A_{2n}^{(2)}$ $a_{ij} = -2 \Rightarrow o(i) = 1$.

Remark 2.

A map o as in notation 1 exists and is:

- i) determined up to a sign, in the untwisted case and in cases $A_{2n}^{(2)}$ and $D_4^{(3)}$;
- ii) uniquely determined, in cases $A_{2n-1}^{(2)}$ and $E_6^{(2)}$.

Definition 3.

Let $\tilde{\psi} = \tilde{\psi}_{X_n^{(k)}} : \tilde{\mathcal{U}}_q^{Dr}(X_n^{(k)}) \rightarrow \mathcal{U}_q^{DJ}(X_n^{(k)})$ be the $\mathbb{C}(q)$ -algebra homomorphism defined on the generators as follows:

$$\begin{aligned} C^{\pm 1} &\mapsto K_{\delta}^{\pm 1}, \quad k_i^{\pm 1} \mapsto K_i^{\pm 1} \quad (i \in I_0), \\ X_{i, \tilde{d}_i r}^+ &\mapsto o(i)^r T_{\lambda_i}^{-r}(E_i), \quad X_{i, \tilde{d}_i r}^- \mapsto o(i)^r T_{\lambda_i}^r(F_i) \quad (i \in I_0, r \in \mathbb{Z}), \\ H_{i, \tilde{d}_i r} &\mapsto \begin{cases} o(i)^r E_{(\tilde{d}_i r \delta, i)} & \text{if } r > 0 \\ o(i)^r F_{(-\tilde{d}_i r \delta, i)} & \text{if } r < 0 \end{cases} \quad (i \in I_0, r \in \mathbb{Z} \setminus \{0\}). \end{aligned}$$

Proposition 4.

- i) $\tilde{\psi}$ is well defined;
- ii) $\tilde{\psi} \circ \tilde{\Omega} = \Omega \circ \tilde{\psi}$;
- iii) $\tilde{\psi} \circ \tilde{t}_i = T_{\lambda_i} \circ \tilde{\psi} \quad \forall i \in I_0$;
- iv) $\tilde{\psi} \circ \tilde{\phi}_i = \varphi_i \circ \tilde{\psi} \quad \forall i \in I_0$;
- v) $\tilde{\psi} \circ \tilde{\Theta} = (\Omega \Xi T_1) \circ \tilde{\psi}$ in cases $A_1^{(1)}$ and $A_2^{(2)}$.

Proof:

i) The relations (ZX^{\pm}) , (CUK) , (CK) and (KX^{\pm}) are obviously preserved by $\tilde{\psi}$; also (XX) (see [Be] and [Da]) and (HXL^{\pm}) hold in

\mathcal{U}_q : it is enough to notice that $\forall i, j \in I_0, \forall r \in \mathbb{Z}$ such that $\max\{\tilde{d}_i, \tilde{d}_j\} | r$ we have $b_{ijr} = (o(i)o(j))^{\tilde{r}} x_{ijr}$ where $\tilde{r} = \frac{r}{\max\{\tilde{d}_i, \tilde{d}_j\}}$ and

$$x_{ijr} = \begin{cases} (o(i)o(j))^r \frac{[ra_{ij}]_{q_i}}{r} & \text{if } k = 1, \text{ or } X_{\tilde{n}}^{(k)} = A_{2n}^{(2)} \text{ and } (i, j) \neq (1, 1) \\ \frac{[2r]_q}{r} (q^{2r} + (-1)^{r-1} + q^{-2r}) & \text{if } (X_{\tilde{n}}^{(k)}, i, j) = (A_{2n}^{(2)}, 1, 1) \\ (o(i)o(j))^{\tilde{r}} \frac{[ra_{ij}^s]_q}{\tilde{r}[d_i]_q} & \text{otherwise} \end{cases}$$

with $a_{ij}^s = \max\{a_{ij}, a_{ji}\}$ (see [Da]).

ii), iii), iv) and v) are trivial.

Theorem 5.

Let $X_{\tilde{n}}^{(k)}$ be different from $A_1^{(1)}$ and $A_2^{(2)}$. Then $\tilde{\psi}$ induces

$$\psi = \psi_{X_{\tilde{n}}^{(k)}} : \mathcal{U}_q^{Dr}(X_{\tilde{n}}^{(k)}) \rightarrow \mathcal{U}_q^{DJ}(X_{\tilde{n}}^{(k)}).$$

Proof: Thanks to corollary §11.19, iii) and to proposition 4, i)-iii) it is enough to prove that $\tilde{\psi}(\mathcal{I}_0^+(S)) = 0$; but this is obvious since $\tilde{\psi}(\mathcal{I}_0^+(S))$ is the ideal generated by the (“positive”) Serre relations.

In order to prove that ψ is well defined also in the remaining cases we propose two different arguments: a direct one, requiring just some simple commutation relations in $\mathcal{U}_q(A_1^{(1)})$ and $\mathcal{U}_q(A_2^{(2)})$ (see lemma 6); and an argument using the injections φ_i (see §3.7).

Lemma 6.

In $\mathcal{U}_q(A_1^{(1)})$ we have that:

i) $E_{\delta+\alpha_1} E_1 = q^2 E_1 E_{\delta+\alpha_1}$.

In $\mathcal{U}_q(A_2^{(2)})$ we have that:

ii) $E_{\delta+\alpha_1} E_1 - q^2 E_1 E_{\delta+\alpha_1} = -[4]_q E_{\delta+2\alpha_1}$;

iii) $E_{\delta+2\alpha_1} E_1 = q^4 E_1 E_{\delta+2\alpha_1}$;

iv) $q^{-3} E_{\delta+\alpha-1} E_1^2 - (q + q^{-1}) E_1 E_{\delta+\alpha-1} E_1 + q^3 E_1^2 E_{\delta+\alpha-1} = 0$.

Proof: i) is an immediate application of the Levendorskii-Soibelman formula (see [LS] and [Da]);

ii) see [Da];

iii) is an immediate application of the Levendorskii-Soibelman formula (see [LS] and [Da]);

iv) follows from ii) and iii).

Theorem 7.

$\tilde{\psi}$ induces $\psi = \psi_{X_{\tilde{n}}^{(k)}} : \mathcal{U}_q^{Dr}(X_{\tilde{n}}^{(k)}) \rightarrow \mathcal{U}_q^{DJ}(X_{\tilde{n}}^{(k)})$.

Proof: Thanks to corollary §11.19, i) and ii), to proposition 4 and to theorem 5 it is enough to notice that $\tilde{\psi}(\mathcal{I}_0^+(X1)) = 0$ in case $A_1^{(1)}$ and

$\tilde{\psi}(\mathcal{I}_0^+(X3^1)) = 0$ in case $A_2^{(2)}$; but this is an immediate consequence of lemma 6, i) and iv).

Otherwise:

Let $h = 1, 2$, $X_{\tilde{n}}^{(k)} = A_4^{(2)}$, $i = \begin{cases} 2 & \text{if } h = 1 \\ 1 & \text{if } h = 2 \end{cases}$ and consider the following well defined diagram:

$$\begin{array}{ccc} \tilde{\mathcal{U}}_q^{Dr}(A_h^{(h)}) & \xrightarrow{\tilde{\psi}_{A_h^{(h)}}} & \mathcal{U}_q^{DJ}(A_h^{(h)}) \\ \downarrow & & \downarrow \varphi_i \\ \mathcal{U}_q^{Dr}(A_h^{(h)}) & \xrightarrow{\phi_i} \mathcal{U}_q^{Dr}(X_{\tilde{n}}^{(k)}) \xrightarrow{\psi_{X_{\tilde{n}}^{(k)}}} & \mathcal{U}_q^{DJ}(X_{\tilde{n}}^{(k)}) \end{array}$$

Without loss of generality we can suppose this diagram to be commutative, by choosing $o_{A_h^{(h)}} : 1 \mapsto o_{X_{\tilde{n}}^{(k)}}(i)$.

Then $\tilde{\psi}_{A_h^{(h)}}$ factors through $\mathcal{U}_q^{Dr}(A_h^{(h)})$ (that is $\psi_{A_h^{(h)}}$ is well defined) because φ_i is injective (see remark §3.7).

Remark 8.

- i) $\psi \circ \Omega = \Omega \circ \psi$;
- ii) $\psi \circ t_i = T_{\lambda_i} \circ \psi \ \forall i \in I_0$;
- iii) $\psi \circ \phi_i = \varphi_i \circ \psi \ \forall i \in I_0$;
- iv) $\psi \circ \Theta = (\Omega \Xi T_1) \circ \psi$ in cases $A_1^{(1)}$ and $A_2^{(2)}$;
- v) $\psi \circ \phi = \varphi$.

Proof:

i)-iv) follow from proposition 4 and from theorem 7.

v) follows from remark §11.9 and theorem 7.

Corollary 9.

ϕ is injective.

Proof: It follows from remark 8,v), since φ is injective (see remark §3.7).

We shall now give a proof of the surjectivity of ψ .

Remark 10.

By the definition of ψ it is obvious that $E_i, F_i, K_i^{\pm 1}$ are in the image of ψ for all $i \in I_0$. Moreover, since $K_\delta^{\pm 1}$ is in the image of ψ , also $K_0^{\pm 1}$ is. But by remark §12.8 $Im(\psi)$ is Ω -stable, so it contains E_0 if and only if it contains F_0 . Then it is enough to prove that $E_0 \in Im(\psi)$.

In the next theorem it will be used that for $(i \in I_0)$ $Im(\psi)$ is a T_{λ_i} -stable subalgebra of \mathcal{U}_q^{DJ} containing $E_j, F_j, K_j^{\pm 1}$ ($j \in I_0, \tilde{j} \in I$) (see remark §12.8); in particular $\mathcal{U}_{q, \pm \alpha}^{DJ, \pm} \subseteq Im(\psi)$ for all $\alpha = \sum_{i \in I_0} m_i \alpha_i$.

Theorem 11.

$\psi : \mathcal{U}_q^{Dr} \rightarrow \mathcal{U}_q^{DJ}$ is surjective.

Proof: Let $\theta = \delta - \alpha_0 = \sum_{i \in I_0} r_i \alpha_i$. Remark that there exists $i \in I_0$ such that either $\tilde{d}_i = r_i = 1$ (recall that θ is a root) or $\tilde{d}_i = 1, r_i = 2, a_{i0} \neq 0$ (so that in particular $\alpha_0 + \alpha_i$ and $\theta - \alpha_i$ are roots). Choose such an $i \in I_0$ and let $\tilde{\theta} = \theta - (r_i - 1)\alpha_i$: $\tilde{\theta}$ is a root.

Let $\lambda_i = \tau_i s_{i_1} \dots s_{i_N}$ (with $(l(\lambda_i) = N, \tau_i \in \mathcal{T})$: then $\lambda_i(\tilde{\theta}) = \tilde{\theta} - \delta < 0$, so that there exists h such that $s_{i_N} \dots s_{i_{h+1}}(\alpha_{i_h}) = \tilde{\theta}$, and we have that $f = T_{i_N}^{-1} \dots T_{i_{r+1}}^{-1}(F_{i_r}) \in \mathcal{U}_{q, -\tilde{\theta}}^{DJ, -} \subseteq Im(\psi)$.

Since $Im(\psi)$ is T_{λ_i} -stable we have that $T_{\lambda_i}(f) = -\tau_i T_{i_1} \dots T_{i_{r-1}}(K_{i_r}^{-1} E_{i_r}) = -K_{\delta - \tilde{\theta}}^{-1} e$ (hence e) belongs to $Im(\psi)$, with $e \in \mathcal{U}_{q, \delta - \tilde{\theta}}^{DJ, +}$.

If $r_i = 1, \delta - \tilde{\theta} = \alpha_0$ and the claim follows ($e = E_0 \in Im(\psi)$).

If $r_i = 2$ then $e \in \mathcal{U}_{q, \alpha_0 + \alpha_i}^{DJ, +}$: remark that if we are not in case $A_{2n}^{(2)}$, since $l(s_i \lambda_i) = l(\lambda_i) + 1$ and $a_{i0} = -1$, $T_i(e) \in \mathcal{U}_{q, \alpha_0}^{DJ, +}$, so that $e = T_i^{-1}(E_0)$; on the other hand in case $A_{2n}^{(2)}$, since $l(s_0 \lambda_i) = l(\lambda_i) - 1$ and $a_{0i} = -1$, $T_0^{-1}(e) \in \mathcal{U}_{q, \alpha_i}^{DJ, +}$, so that $e = T_0(E_i)$. In both cases $e = -[E_0, E_i]_{q_i}^{a_{i0}}$. Commuting e with $F_i (\in Im(\psi))$ we get $Im(\psi) \ni [-[E_0, E_i]_{q_i}^{a_{i0}}, F_i] = q_i^{a_{i0}} [[E_i, F_i], E_0]_{q_i}^{-a_{i0}} = [a_{i0}]_{q_i} K_i E_0$, which concludes the proof.

Proving that ψ is injective requires further analysis.

§13. APPENDIX: NOTATIONS.

In this appendix, in order to make it easier for the reader to follow the exposition, most of the notations defined in the paper are collected, with the indication of the point where they are introduced and eventually characterized.

The present list includes neither the notations related to the definition and the structure of the Drinfeld-Jimbo presentation of the quantum algebras, since they are all given synthetically in section §3 where they can be easily consulted, nor the notations introduced in definition §4.2, because there is no reference to them outside section §4.

Also the relations listed in proposition §4.25 are not redefined in this appendix, but for some of them other descriptions proposed and used throughout the paper are here recalled.

Analogously it seems useful to gather the remaining notations, which are spread out in the paper.

Dynkin diagrams, root and weight lattices:

$\tilde{\Gamma} =$ (indecomposable) Dynkin diagram of finite type	§1
$\tilde{I} =$ set of vertices of $\tilde{\Gamma}$	§1
$\tilde{n} = \#\tilde{I}$	§1
$\tilde{A} =$ Cartan matrix of $\tilde{\Gamma}$	§1
$\chi =$ automorphism of $\tilde{\Gamma}$	§1
$k = o(\chi)$	§1
$I_0 = \tilde{I}/\chi$	§1
$n = \#I_0$	§1
$\bar{\cdot} : \tilde{I} \rightarrow I_0$ natural projection	§1
$\tilde{\cdot} : I_0 \rightarrow \tilde{I}$ section ($a_{ij} \neq 0 \Rightarrow \tilde{a}_{i,\tilde{j}} \neq 0$)	§4.4, (§4.10)
$I = I_0 \cup \{0\}$	§1
$\Gamma =$ Dynkin diagram of affine type with set of vertices I	§1
$\Gamma_0 =$ Dynkin subdiagram of Γ with set of vertices I_0	§1
$A = (a_{ij})_{i,j \in I}$ Cartan matrix of Γ	§1
$A_0 = (a_{ij})_{i,j \in I_0}$ Cartan matrix of Γ_0	§1
$d_i : \min\{d_i i \in I\} = 1, \text{diag}(d_i i \in I)A$ symmetric	§2
$\tilde{d}_i = \begin{cases} 1 & \text{if } k = 1 \text{ or } X_{\tilde{n}}^{(k)} = A_{2n}^{(2)} \\ d_i & \text{otherwise;} \end{cases}$	§2
$\tilde{d}_{i,j} = \max\{\tilde{d}_i, \tilde{d}_j\}$	§4.8
$\tilde{d} = \max\{\tilde{d}_i i \in I_0\}$	§4.1
$\tilde{Q} = \mathbb{Z}^{\tilde{I}} = \oplus_{i' \in \tilde{I}} \mathbb{Z}\tilde{\alpha}_{i'}$	§1
$(\tilde{\alpha}_{i'} \tilde{\alpha}_{j'}) : W - \text{invariant}, (\tilde{\alpha}_{i'} \tilde{\alpha}_{i'}) = \frac{2k}{\tilde{d}} \text{ if } \tilde{\alpha}_{i'} \text{ is short}$	§4.1
$Q = \mathbb{Z}^I = \oplus_{i \in I} \mathbb{Z}\alpha_i$	§1
$Q_0 = \oplus_{i \in I_0} \mathbb{Z}\alpha_i$	§1
$(\alpha_i \alpha_j) = d_i a_{ij}$	§2

$$\delta : \quad \delta \in Q, \delta - \alpha_0 \in Q_0, (\delta|Q) = 0 \quad \S 2$$

$$r_i : \quad \delta = \sum_{i \in I} r_i \alpha_i \quad \S 1, \S 2$$

$$\theta = \delta - \alpha_0 \quad \S 1$$

$$\lambda_i : \quad (\lambda_i|\alpha_j) = \tilde{d}_i \delta_{ij} \quad (i, j \in I_0) \quad \S 2$$

Other notations:

$$\omega = \text{primitive } k^{\text{th}} \text{ root of } 1 \quad \S 4.1$$

$$q_i = q^{d_i} \quad \S 3.1$$

$$I_{\mathbb{Z}} = \{(i, r) \in I_0 \times \mathbb{Z} | \tilde{d}_i | r\} \quad \S 4.5$$

$$\varepsilon = \pm 1 \quad \S 4.2, \S 4.25$$

$$b_{ijr} = \begin{cases} 0 & \text{if } \tilde{d}_{i,j} \nmid r \\ \frac{[2r]_q(q^{2r} + (-1)^{r-1} + q^{-2r})}{r} & \text{if } (X_{\tilde{n}}^{(k)}, i, j) = (A_{2n}^{(2)}, 1, 1) \\ \frac{[\tilde{r}a_{ij}]_{q_i}}{\tilde{r}} & \text{otherwise, with } \tilde{r} = \frac{r}{\tilde{d}_{i,j}}. \end{cases} \quad \S 4.14$$

$$[a, b]_u = ab - uba \quad \S 4.16$$

$$\mathbf{1} = \mathbf{1}_l = (1, \dots, 1) \in \mathbb{Z}^l \quad \S 8.1$$

$$\{e_1, \dots, e_l\} = \text{canonical basis of } \mathbb{Z}^l \quad \S 8.1$$

$$\overline{(r_1, \dots, r_l)} = (r_l, \dots, r_1) \quad \S 8.1$$

$$o : \quad I_0 \rightarrow \{\pm 1\}, \quad a_{ij} \neq 0 \Rightarrow o(i)o(j) = -1, \quad \S 12.1$$

$$\text{if } k \neq 1 \text{ and } X_{\tilde{n}}^{(k)} \neq A_{2n}^{(2)} \quad a_{ij} = -2 \Rightarrow o(i) = 1$$

Generators of the $\mathbb{C}(q)$ -algebras:

$$\mathcal{G} = \{C^{\pm 1}, \mathcal{K}_{i',r}^{\pm 1}, \mathcal{X}_{i',r}^{\pm}, \mathcal{H}_{i',s} | i' \in \tilde{I}, r, s \in \mathbb{Z}, s \neq 0\} \quad \S 4.2$$

$$G = \{C^{\pm 1}, K_i^{\pm 1}, X_{i,r}^{\pm}, H_{i,s} | i \in I_0, r, s \in \mathbb{Z}, s \neq 0\} \quad \S 4.6, \S 4.25$$

$$G' = \{C^{\pm 1}, K_i^{\pm 1}, X_{i,r}^{\pm}, H_{i,s} | i \in I_0, r, s \in \tilde{d}_i \mathbb{Z}, s \neq 0\} \quad \S 4.6, \S 4.26$$

$$\bar{G} = \{C^{\pm 1}, K_i^{\pm 1}, X_{i,r}^{\pm} | i \in I_0, r \in \tilde{d}_i \mathbb{Z}\} \quad \S 6.1$$

$$G^+ = \{X_{i,r}^+ | (i, r) \in I_0 \times \mathbb{Z}\} \quad \S 6.11$$

$$G^- = \{X_{i,r}^- | (i, r) \in I_0 \times \mathbb{Z}\} \quad \S 6.11$$

$$G'^+ = \{X_{i,r}^+ | (i, r) \in I_{\mathbb{Z}}\} \quad \S 6.12$$

$$G'^- = \{X_{i,r}^- | (i, r) \in I_{\mathbb{Z}}\} \quad \S 6.12$$

Relations in the $\mathbb{C}(q)$ -algebras:

$$(\mathcal{DR}) = (\mathcal{Z}, \mathcal{ZX}, \mathcal{C}, \mathcal{KK}, \mathcal{KX}, \mathcal{KH}, \mathcal{XX}, \mathcal{HX}, \mathcal{HH}, \mathcal{XFG}, \mathcal{X}3^\varepsilon, \mathcal{S}, \mathcal{XP}) \quad \S 4.2$$

$$(DR) = (ZX, ZH, CUK, CK, KX, KH, XX, HX, HH, XD, X1, X2, X3^\varepsilon, SUL, S2, S3) \quad \S 4.25$$

$$(ZX^\pm), (ZH) \quad \S 4.6$$

$$(HX^\pm) : [H_{i,r}, X_{j,s}^\pm] = \begin{cases} 0 & \text{if } \tilde{d}_j \nmid r \\ \pm b_{ijr} C^{\frac{r \pm |r|}{2}} X_{j,r+s}^\pm & \text{if } \tilde{d}_j | r \end{cases} \quad \S 4.26$$

$$(XD^\pm) : M_{(2)}^\pm((i, \tilde{d}_i r), (j, \tilde{d}_j s)) = 0 \quad \S 5.14$$

$$(X1^\pm) : \sum_{\sigma \in \mathcal{S}_2} \sigma.M_i^\pm(\tilde{d}_i r) = 0 \quad \S 5.14$$

$$(X2^\pm) : \sum_{\sigma \in \mathcal{S}_2} \sigma.M_{(2,2)}^\pm(r) = 0 \quad \S 5.14$$

$$(X3^{\varepsilon, \pm}) : \sum_{\sigma \in \mathcal{S}_3} \sigma.M_{(3)}^{\varepsilon, \pm}(r) = 0 \quad \S 5.14$$

$$(S(UL)^\pm) : \sum_{\sigma \in \mathcal{S}_{1-a_{ij}}} \sigma.X_{i,j;1-a_{ij};1}^\pm(r; s) = 0 \quad \S 5.3, \S 5.9$$

$$\sum_{\sigma \in \mathcal{S}_{1-a_{ij}}} \sigma.M_{i,j;1-a_{ij};1}^\pm(r; s) = 0 \quad \S 5.5, \S 5.10$$

$$\sum_{\sigma \in \mathcal{S}_{1-a_{ij}}} \sigma.M_{i,j;1-a_{ij};1}^\pm(\tilde{d}_i r; \tilde{d}_j s) = 0 \quad \S 5.14$$

$$\sum_{\sigma \in \mathcal{S}_{1-a_{ij}}} \sigma.X_{i,j;1-a_{ij};1}^{\pm}(\tilde{d}_i r; \tilde{d}_j s) = 0 \quad \S 5.14$$

$$(S2^+) : \sum_{\sigma \in \mathcal{S}_2} \sigma.((q^2 + q^{-2})[[X_{j,s}^+, X_{i,r_1+1}^+]_{q^2}, X_{i,r_2}^+] + \quad \S 5.6$$

$$+ q^2[[X_{i,r_1+1}^+, X_{i,r_2}^+]_{q^2}, X_{j,s}^+]_{q^{-4}}) = 0$$

$$\sum_{\sigma \in \mathcal{S}_2} \sigma.((q^2 + q^{-2})[[X_{j,s}^+, X_{i,r_1}^+]_{q^{-2}}, X_{i,r_2+1}^+] + \quad \S 5.6$$

$$+ [X_{j,s}^+, [X_{i,r_2+1}^+, X_{i,r_1}^+]_{q^2}]_{q^{-4}}) = 0$$

$$\sum_{\sigma \in \mathcal{S}_2} \sigma.([X_{j,s}^+, X_{i,r_1+1}^+]_{q^{-2}}, X_{i,r_2}^+] + \quad \S 5.6$$

$$- q^2[X_{i,r_1+1}^+, [X_{j,s}^+, X_{i,r_2}^+]_{q^{-2}}]_{q^{-4}}) = 0$$

$$(S3^{\pm}) : \sum_{\sigma \in \mathcal{S}_2} \sigma.((q^2 + q^{-4})[[X_{j,s}^{\pm}, X_{i,r_1 \pm 2}^{\pm}]_{q^3}, X_{i,r_2}^{\pm}]_{q^{-1}} + \quad \S 5.6$$

$$+ (1 - q^{-2} + q^{-4})[[X_{j,s}^{\pm}, X_{i,r_1 \pm 1}^{\pm}]_{q^3}, X_{i,r_2 \pm 1}^{\pm}]_q +$$

$$+ q^2[[X_{i,r_1 \pm 2}^{\pm}, X_{i,r_2}^{\pm}]_{q^2} + [X_{i,r_2 \pm 1}^{\pm}, X_{i,r_1 \pm 1}^{\pm}]_{q^2}, X_{j,s}^{\pm}]_{q^{-6}}) = 0$$

$$(Sk^{\pm}) : \sum_{\sigma \in \mathcal{S}_2} \sigma. \sum_{\substack{u,v \geq 0 \\ u+v=-1-a_{ij}}} q^{v-u} X_{i,j;2;-a_{ij}}^{\pm}(r_1 \pm v, r_2 \pm u; s) = 0 \quad \S 5.3$$

$$\sum_{\sigma \in \mathcal{S}_2} \sigma.X_{[k]}^{\pm}(r; \tilde{d}s) = 0 \quad \S 5.14$$

$$(T2^{\pm}) : \sum_{\sigma \in \mathcal{S}_2} \sigma.([X_{j,s}^{\pm}, X_{i,r_1 \pm 1}^{\pm}]_{q^2}, X_{i,r_2}^{\pm}) = 0 \quad \S 5.7$$

$$\sum_{\sigma \in \mathcal{S}_2} \sigma.M_{[2]}^{\pm}(r; 2s) = 0 \quad \S 5.14$$

$$(T3^{\pm}) : \sum_{\sigma \in \mathcal{S}_2} \sigma.((q^2 + 1)[[X_{j,s}^{\pm}, X_{i,r_1 \pm 2}^{\pm}]_{q^3}, X_{i,r_2}^{\pm}]_{q^{-1}} + \quad \S 5.7$$

$$+ [[X_{j,s}^{\pm}, X_{i,r_1 \pm 1}^{\pm}]_{q^3}, X_{i,r_2 \pm 1}^{\pm}]_q) = 0$$

$$\sum_{\sigma \in \mathcal{S}_2} \sigma.M_{[3]}^{\pm}(r; 3s) = 0 \quad \S 5.14$$

$$(HXL^{\pm}) : [H_{i,r}, X_{j,s}^{\pm}] = \pm b_{ijr} C^{\frac{r \mp |r|}{2}} X_{j,r+s}^{\pm} \quad (\tilde{d}_i \leq |r| \leq \tilde{d}_{ij}) \quad \S 6.1$$

$$(XXD) : [X_{i,r}^+, X_{j,s}^-] = 0 \quad (i \neq j) \quad \S 7.4$$

$$(XXE) : [X_{i,r}^+, X_{i,-r}^-] = \frac{C^r k_i - C^{-r} k_i^{-1}}{q_i - q_i^{-1}} \quad \S 7.4$$

$$(XXH^+) : [X_{i,r}^+, X_{i,s}^-] = \frac{C^{-s} k_i \tilde{H}_{i,r+s}^+}{q_i - q_i^{-1}} \quad (r + s > 0) \quad \S 7.4$$

$$(XXH^-) : [X_{i,r}^+, X_{i,s}^-] = -\frac{C^{-r} \tilde{H}_{i,r+s}^- k_i^{-1}}{q_i - q_i^{-1}} \quad (r + s < 0) \quad \S 7.4$$

$\mathbb{C}(q)$ -algebras:

\mathcal{U}_q = Drinfeld and Jimbo quantum algebra	§3.2
$\mathcal{U}_q^{DJ} = \mathcal{U}_q(\Gamma)$	§3.2
$\mathcal{U}_q^{fin} = \mathcal{U}_q(\Gamma_0)$	§3.2
$\mathcal{U}_q^{Dr} = (\mathcal{G} \mathcal{DR})$	§4.2
$\mathcal{U}_q^{Dr} = (G DR)$	§4.25, §4.26
$\tilde{\mathcal{U}}_q^{Dr} = (G ZX^\pm, CUK, CK, KX^\pm, XX, HXL^\pm)$	§6.1
$\bar{\mathcal{U}}_q^{Dr} = (\bar{G} ZX^\pm, CUK, CK)$	§6.1
$\mathcal{F}_q^\pm = (G^\pm ZX^\pm) = (G'^\pm)$	§6.11, §6.12

Elements in the $\mathbb{C}(q)$ -algebras:

$$\tilde{H}_{i,\pm r}^\pm (H_{i,r}) : \sum_{r \in \mathbb{Z}} \tilde{H}_{i,\pm r}^\pm u^r = \exp\left(\pm (q_i - q_i^{-1}) \sum_{r > 0} H_{i,\pm r} u^r\right) \quad \S 4.25 \quad (\S 6.6)$$

$$\tilde{H}_{i,\pm r}^\pm = \begin{cases} (q_i - q_i^{-1}) k_i^{-1} [X_{i,r}^+, X_{i,0}^-] & \text{if } r, \pm r > 0 \\ (q_i - q_i^{-1}) [X_{i,-r}^-, X_{i,0}^+] k_i & \text{if } r > 0, \pm r < 0 \\ 1 & \text{if } r = 0 \\ 0 & \text{if } r < 0, \end{cases} \quad \S 6.6$$

$$X_{i,j;l;a}(r; s) = \sum_{u=0}^l (-1)^u \begin{bmatrix} l \\ u \end{bmatrix}_{q_i^a} X_{i,r_1}^\pm \cdots X_{i,r_u}^\pm X_{j,s}^\pm X_{i,r_{u+1}}^\pm \cdots X_{i,r_l}^\pm \quad \S 5.2$$

$$M_{i,j;l;a}(r; s) = \begin{cases} X_{j,s}^\pm & \text{if } l = 0 \\ [M_{i,j;l-1;a}(r_1, \dots, r_{l-1}; s), X_{i,r_l}^\pm]_{q_i^{-a_{ij}-2a(l-1)}} & \text{if } l > 0 \end{cases} \quad \S 5.4$$

$$M_2^\pm((i, r), (j, s)) = [X_{i,r \pm \tilde{d}_{ij}}^\pm, X_{j,s}^\pm]_{q_i^{a_{ij}}} + [X_{j,s \pm \tilde{d}_{ij}}^\pm, X_{i,r}^\pm]_{q_j^{a_{ji}}} \quad \S 5.12$$

$$M_i^\pm(r) = [X_{i,r_1 \pm \tilde{d}_i}^\pm, X_{i,r_2}^\pm]_{q_i^2} \quad \S 5.12$$

$$M_{(2,2)}^\pm(r) = [X_{1,r_1 \pm 2}^\pm, X_{1,r_2}^\pm]_{q^2} - q^4 [X_{1,r_1 \pm 1}^\pm, X_{1,r_2 \pm 1}^\pm]_{q^{-6}} \quad \S 5.12$$

$$M_{(3)}^{\varepsilon,\pm}(r) = [[X_{1,r_1 \pm \varepsilon}^\pm, X_{1,r_2}^\pm]_{q^{2\varepsilon}}, X_{1,r_3}^\pm]_{q^{4\varepsilon}} \quad \S 5.12$$

$$X_{[k]}^\pm(r; s) = \sum_{\substack{u,v \geq 0 \\ u+v=k-1}} q^{v-u} X_{i,j;2;k}^\pm(r_1 \pm u, r_2 \pm v; s) \quad \S 5.12$$

$$M_{[2]}^\pm(r; s) = M_{i,j;2;1}^\pm(r_1 \pm 1, r_2; s) \quad \S 5.12$$

$$M_{[3]}^\pm(r; s) = (q^2 + 1) M_{i,j;2;2}^\pm(r_1 \pm 2, r_2; s) + M_{i,j;2;1}^\pm(r_1 \pm 1, r_2 \pm 1; s) \quad \S 5.12$$

$$k_\alpha : k_{m\delta + \sum_{i \in I_0} m_i \alpha_i} = C^m \prod_{i \in I_0} k_i^{m_i} \quad \S 6.9$$

Relations and ideals:

a) given the relations

$$(R) \quad S_\zeta(r, s) = 0 \quad (\zeta \in \mathcal{Z}, \quad r \in \mathbb{Z}^l, \quad s \in \mathbb{Z}^{\bar{l}})$$

denote by (R_ζ) ($\zeta \in \mathcal{Z}$) the relations

$$(R_\zeta) \quad S_\zeta(r, s) = 0 \quad (r \in \mathbb{Z}^l, \quad s \in \mathbb{Z}^{\bar{l}}); \quad \S 5.1$$

b) given the relations

$$(R^\pm) \quad S_\zeta^\pm(r, s) = 0 \quad (\zeta \in \mathcal{Z}, \quad r \in \mathbb{Z}^l, \quad s \in \mathbb{Z}^{\bar{l}})$$

denote by (R) the relations

$$(R) \quad S_{\zeta'}(r, s) = 0 \quad (\zeta' \in \mathcal{Z} \times \{\pm\}, \quad r \in \mathbb{Z}^l, \quad s \in \mathbb{Z}^{\bar{l}}) \quad \S 5.1$$

where $S_{(\zeta, \pm)} = S_\zeta^\pm$;

c) given relations (R) as in a), denote by $\mathcal{I}(R)$ the ideal

$$\mathcal{I}(R) = (S_\zeta(r, s) | \zeta \in \mathcal{Z}, \quad r \in \mathbb{Z}^l, \quad s \in \mathbb{Z}^{\bar{l}}), \quad \S 5.1$$

by $\mathcal{I}_{cte}(R)$ the ideal

$$\mathcal{I}_{cte}(R) = (S_\zeta(r \mathbf{1}_l, s) | \zeta \in \mathcal{Z}, \quad r \in \mathbb{Z}, \quad s \in \mathbb{Z}^{\bar{l}}), \quad \S 9.1$$

by $\mathcal{I}_0(R)$ the ideal

$$\mathcal{I}_0(R) = (S_\zeta(\underline{0}) | \zeta \in \mathcal{Z}) \quad \S 9.1$$

and, if $\zeta \in \mathcal{Z}$, $r \in \mathbb{Z}^l$, $s \in \mathbb{Z}^{\bar{l}}$, denote by $\mathcal{I}_{(r,s)}(R_\zeta)$ the ideal

$$\mathcal{I}_{(r,s)}(R_\zeta) = (S_\zeta(r \mathbf{1}_l, s)); \quad \S 9.1$$

d) given relations (R^\pm) as in a), denote by $\mathcal{I}_*^\pm(R)$ the ideals

$$\mathcal{I}_*^+(R) = \mathcal{I}_*(R^+), \quad \mathcal{I}_*^-(R) = \mathcal{I}_*(R^-) \quad (* \in \{\emptyset, cte, 0\}); \quad \S 5.1, \S 9.1$$

e) given a family of relations $({}^h R)$ as in a) denote by $\mathcal{I}_*({}^{(1)}R, \dots, {}^{(m)}R)$ the ideals

$$\mathcal{I}_*({}^{(1)}R, \dots, {}^{(m)}R) = (\mathcal{I}_*({}^{(1)}R), \dots, \mathcal{I}_*({}^{(m)}R)) \quad (* \in \{\emptyset, cte, 0\}). \quad \S 5.1, \S 9.1$$

(Anti)homomorphisms:

$$\begin{aligned}
\bar{\Omega}, \tilde{\Omega}, \Omega : q &\mapsto q^{-1}, C^{\pm 1} \mapsto C^{\mp 1}, k_i^{\pm 1} \mapsto k_i^{\mp 1}, X_{i,r}^{\pm} \mapsto X_{i,-r}^{\mp} && \S 6.15, \S 7.6, \S 8.2 \\
\Theta_{\mathcal{F}}^{+} : q &\mapsto q^{-1}, X_{i,r}^{+} \mapsto X_{i,-r}^{+} && \S 6.15 \\
\Theta_{\mathcal{F}}^{-} : q &\mapsto q^{-1}, X_{i,r}^{-} \mapsto X_{i,-r}^{-} && \S 6.15 \\
\bar{\Theta}, \tilde{\Theta}, \Theta : q &\mapsto q^{-1}, C^{\pm 1} \mapsto C^{\pm 1}, k_i^{\pm 1} \mapsto k_i^{\mp 1}, && \S 6.15, \S 7.13, \S 8.4 \\
X_{i,r}^{+} &\mapsto -X_{i,-r}^{+} k_i C^{-r}, X_{i,r}^{-} \mapsto -k_i^{-1} C^{-r} X_{i,-r}^{-} \\
\bar{t}_i, \tilde{t}_i, t_i : C^{\pm 1} &\mapsto C^{\pm 1}, k_j^{\pm 1} \mapsto (k_j C^{-\delta_{ij} \tilde{d}_i})^{\pm 1}, X_{j,r}^{\pm} \mapsto X_{j, r \mp \delta_{ij} \tilde{d}_i}^{\pm} && \S 6.15, \S 7.10, \S 8.6 \\
\bar{\phi}_i, \tilde{\phi}_i, \phi_i : q &\mapsto q_i, C^{\pm 1} \mapsto C^{\pm \tilde{d}_i}, k_i^{\pm 1} \mapsto k_i^{\pm 1}, X_r^{\pm} \mapsto X_{i, \tilde{d}_i r}^{\pm} && \S 6.15, \S 7.2 \\
\phi : K_i^{\pm 1} &\mapsto k_i^{\pm 1}, E_i \mapsto X_{i,0}^{+}, F_i \mapsto X_{i,0}^{-} (i \in I_0) && \S 11.8 \\
\tilde{\psi}, \psi : C^{\pm 1} &\mapsto K_{\delta}^{\pm 1}, k_i^{\pm 1} \mapsto K_i^{\pm 1}, && \S 12.3, \S 12.5, \S 12.7 \\
X_{\tilde{d}_i r}^{+} &\mapsto o(i)^r T_{\lambda_i}^{-r}(E_i), X_{\tilde{d}_i r}^{-} \mapsto o(i)^r T_{\lambda_i}^r(F_i)
\end{aligned}$$

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